

Improved light quark masses from pseudoscalar sum rules

Stephan Narison^a

^aLaboratoire Univers et Particules de Montpellier, CNRS-IN2P3, Case 070, Place Eugène Bataillon, 34095 - Montpellier, France.

Abstract

Using ratios of the inverse Laplace transform sum rules within stability criteria for the subtraction point μ in addition to the ones of the usual τ spectral sum rule variable and continuum threshold t_c , we extract the $\pi(1300)$ and $K(1460)$ decay constants to order α_s^4 of perturbative QCD by including power corrections up to dimension-six condensates, tachyonic gluon mass, instanton and finite width corrections. Using these inputs with enlarged generous errors, we extract, *in a model-independent and conservative ways*, the sum of the scale-independent renormalization group invariant (RGI) quark masses ($\hat{m}_u + \hat{m}_q$) : $q \equiv d, s$ and the corresponding running masses ($\bar{m}_u + \bar{m}_q$) evaluated at 2 GeV. By giving the value of the ratio m_u/m_d , we deduce the running quark masses $\bar{m}_{u,d,s}$ and condensate $\langle \bar{u}u \rangle$ and the scale-independent mass ratios : $2m_s/(m_u + m_d)$ and m_s/m_d . Using the positivity of the QCD continuum contribution to the spectral function, we also deduce, from the inverse Laplace transform sum rules, for the first time to order α_s^4 , new lower bounds on the RGI masses which are translated into the running masses at 2 GeV and into upper bounds on the running quark condensate $\langle \bar{u}u \rangle$. Our results summarized in Table 3 and compared with our previous results and with recent lattice averages suggest that precise phenomenological determinations of the sum of light quark masses require improved experimental measurements of the $\pi(1.3)$ and $K(1.46)$ hadronic widths and/or decay constants which are the dominant sources of errors in the analysis.

Keywords: QCD spectral sum rules, meson decay constants, light quark masses, chiral symmetry.

1. Introduction and a short historical overview

Pseudoscalar sum rules have been introduced for the first time in [1] for giving a bound on the sum of running light quark masses defined properly for the first time in the \overline{MS} -scheme by [2]. Its Laplace transform version including power corrections introduced by SVZ [3] ^{1,2} has been applied few months later to the pseudoscalar channel in [9] and extended to the estimate of the SU(3) corrections to kaon PCAC in [10]. Its first application to the scalar channel was in [11]. Later on, the previous analysis has been reconsidered in [12] for extracting e.g. the $\pi(1300)$ and $K(1460)$ decay constants. The first FESR analysis in the pseudoscalar channel has been done in [13, 14] which has been used later on by various authors ³.

However, the light pseudoscalar channel is *quite delicate* as the PT radiative corrections ([1, 15] for the α_s , [13, 16] for the α_s^2 , [17] for the α_s^3 and [18] for the α_s^4 corrections) are quite large for low values of $Q^2 \approx 1 \text{ GeV}^2$ where the Goldstone pion contribution is expected to dominate the spectral function, while (less controlled) and controversial instanton-like contributions [19–21] ⁴ might break the operator product expansion

(OPE) at a such low scale. However, working at higher values of Q^2 for avoiding these QCD series convergence problems, one has to face the dominant contribution from radial excited states where a little experimental information is known. Some models have been proposed in the literature for parametrizing the high-energy part of the spectral function. It has been proposed in [12] to extract the $\pi(1300)$ and $K(1460)$ decay constants by combining the pseudoscalar and scalar sum rules which will be used in the Laplace sum rules for extracting the light quark masses. Though interesting, the analysis was quite qualitative (no estimate of the errors) such that it is not competitive for an accurate determination of the quark masses. This estimate has been improved in [6] using a narrow width approximation (NWA). Later on, a much more involved ChPT based parametrization of the pion spectral function has been proposed in [23] where the model dependence mainly appears in the interference between the $\pi(1300)$ and the $\pi(1800)$. Using FESR with some weight functions inspired from τ -decay [25–30], the authors of [31] have extracted the decay constants of the $\pi(1300)$ and the $\pi(1800)$ by assuming that they do not interfere in the spectral function. The results for the spectral function are one of the main ingredient for extracting the light quark masses from pseudoscalar sum rules and it is important to have a good control (and a model-independence) of its value for a more precise and model-independent determination of such light quark masses.

In this paper, our aim is to extract the spectral function or the $\pi(1300)$ and $K(1460)$ decay constants from the ratio of Laplace sum rules known to order α_s^4 of perturbation theory (PT) and including power corrections up to dimension six within the SVZ

Email address: snarison@yahoo.fr (Stephan Narison)

¹For review, see e.g. [4–8].

²Radiative corrections to the exponential sum rules have been first derived in [9], where it has been noticed that the PT series has the property of an Inverse Laplace transform.

³For reviews, see e.g.: [6, 7].

⁴However, analogous contribution might lead to some contradiction in the scalar channel [22].

expansion plus those beyond it such as the tachyonic gluon mass and the instanton contributions. With this result, we shall extract the light quark mass values at the same approximation of the QCD series.

2. The pseudoscalar Laplace sum rule

• The form of the sum rules

We shall be concerned with the two-point correlator :

$$\psi_5^P(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T J_5^P(x) J_5^P(0)^\dagger | 0 \rangle, \quad (1)$$

where $J_5^P(x)$ is the local pseudoscalar current :

$$J_5^P(x) \equiv (m_u + m_q) \bar{u}(i\gamma_5)q, \quad q = d, s; P = \pi, K. \quad (2)$$

The associated Laplace sum rules (LSR) $\mathcal{L}_5^P(\tau)$ and its ratio $\mathcal{R}_5^P(\tau)$ read [3]⁵:

$$\mathcal{L}_5^P(\tau, \mu) = \int_{(m_u+m_q)^2}^{\tau_c} dt e^{-t\tau} \frac{1}{\pi} \text{Im} \psi_5^P(t, \mu), \quad (3)$$

$$\mathcal{R}_5^P(\tau, \mu) = \frac{\int_{(m_u+m_q)^2}^{\tau_c} dt t e^{-t\tau} \frac{1}{\pi} \text{Im} \psi_5^P(t, \mu)}{\int_{(m_u+m_q)^2}^{\tau_c} dt e^{-t\tau} \frac{1}{\pi} \text{Im} \psi_5^P(t, \mu)}, \quad (4)$$

where μ is the subtraction point which appears in the approximate QCD series. The ratio of sum rules $\mathcal{R}_5^P(\tau, \mu)$ is useful here for extracting the contribution of the radial excitation P' to the spectral function, while the Laplace sum rule $\mathcal{L}_5^P(\tau, \mu)$ will be used for determining the sum of light quark masses.

• The QCD expression within the SVZ expansion

As mentioned earlier, the perturbative expression of the two-point correlator $\psi_5^P(q^2)$ is known up to order α_s^4 from successive works [1, 13, 16–18]. For a convenience of the reader, we give below the numerical expression⁶:

$$\begin{aligned} \mathcal{L}_5^P(\tau) &= \frac{3}{8\pi^2} (\bar{m}_u + \bar{m}_q)^2 \tau^{-2} \left[1 + \sum_{n=1,4} \delta_n^{(0)} a_s^n - \right. \\ &\quad \left. 2m_q^2 \tau \left(1 + \sum_{n=1,2} \delta_n^{(2)} a_s^n \right) + \right. \\ &\quad \left. \tau^2 \delta^{(4)} + \tau^3 \delta^{(6)} \right], \end{aligned} \quad (5)$$

where \bar{m}_q is the running quark mass evaluated at the scale μ . From the analytic expression compiled in [33], we derive the

⁵A quantum mechanics interpretation of these Laplace sum rules has been given by [32].

⁶In the following, we shall not expand the QCD expression: $1/(1 + ka_s + \dots + (np)\tau + \dots)$ as: $1 - ka_s + k^2 a_s^2 + \dots - (np)\tau + (np)^2 \tau^2 + \dots$, but keep its non-expanded form.

numerical PT corrections:

$$\begin{aligned} \delta_1^{(0)} &= 4.82107 - 2l_\mu, \\ \delta_2^{(0)} &= 21.976 - 28.0729l_\mu + \frac{17}{4}l_\mu^2, \\ \delta_3^{(0)} &= 53.1386 - 677.987l_\mu + 102.82l_\mu^2 - \frac{221}{24}l_\mu^3, \\ \delta_4^{(0)} &= -31.6283 + 756701l_\mu + 1231.57l_\mu^2 - \\ &\quad 321.968l_\mu^3 + \frac{7735}{384}l_\mu^4, \\ \delta_1^{(2)} &= 7.64213 - 4l_\mu, \\ \delta_2^{(2)} &= 51.0915 - 62.93l_\mu + \frac{25}{2}l_\mu^2, \end{aligned} \quad (6)$$

where : $a_s \equiv \alpha_s/\pi$; $l_\mu \equiv -\text{Log}(\tau\mu^2)$. The non-perturbative corrections are combinations of RGI quantities defined in [26, 34, 35]:

$$\begin{aligned} \overline{m_q \langle \bar{q}q \rangle} &= m_q \langle \bar{q}q \rangle + \frac{3}{7\pi^2} m_q^4 \left(\frac{1}{a_s} - \frac{53}{24} \right) \\ \overline{\langle \alpha_s G^2 \rangle} &= \langle \alpha_s G^2 \rangle \left(1 + \frac{16}{9} a_s \right) - \frac{16}{9} \alpha_s \left(1 + \frac{91}{24} a_s \right) m_q \langle \bar{q}q \rangle, \\ &\quad - \frac{1}{3\pi} \left(1 + \frac{4}{3} a_s \right) m_q^4. \end{aligned} \quad (7)$$

In terms of these quantities, they read [1, 6, 36, 37]:

$$\delta^{(4)} = \frac{4\pi^2}{3} (\delta_q^{(4)} + \delta_g^{(4)}), \quad (8)$$

with:

$$\begin{aligned} \delta_q^{(4)} &= -2m_q \langle \bar{u}u \rangle \left[1 + a_s (5.821 - 2l_\mu) \right] + \\ &\quad \overline{m_q \langle \bar{q}q \rangle} \left[1 + a_s (5.266 - 2l_\mu) \right] - \\ &\quad \frac{3}{7\pi^2} m_q^4 \left(\frac{1}{a_s} + 2.998 - \frac{15}{4} l_\mu \right), \\ \delta_g^{(4)} &= \frac{1}{4\pi} \overline{\langle \alpha_s G^2 \rangle} \left[1 + a_s (4.877 - 2l_\mu) \right]. \end{aligned} \quad (9)$$

The contribution of the $d = 6$ condensate is:

$$\delta^{(6)} = -\frac{4\pi^2}{3} \left[m_q \langle \bar{u}Gu \rangle + \frac{32}{27} \pi \rho \alpha_s (\langle \bar{u}u \rangle^2 + \langle \bar{q}q \rangle^2 - 9 \langle \bar{u}u \rangle \langle \bar{q}q \rangle) \right], \quad (10)$$

where $\langle \bar{u}Gu \rangle \equiv \langle \bar{u}(\lambda^a/2)G_a^{\mu\nu} \sigma_{\mu\nu} u \rangle \equiv M_0^2 \langle \bar{u}u \rangle$ with $M_0^2 = (0.8 \pm 0.2) \text{ GeV}^2$ [38–40] is the quark-gluon mixed condensate; $\rho = (4.2 \pm 1.3)$ [28, 38, 41] indicates the violation of the vacuum saturation assumption of the four-quark operators.

• Tachyonic gluon mass and estimate of larger order PT-terms

The tachyonic gluon mass λ of dimension two has been introduced in [42, 43] and appears naturally in most holographic QCD models [44]. Its contribution is “dual” to the uncalculated higher order terms of the PT series [45] and disappears for long PT series like in the case of lattice calculations [46], but should

remain when only few terms of the PT series are calculated like in the case studied here. Its contribution reads [43]:

$$\mathcal{L}_5^P(\tau)|^{tach} = -\frac{3}{2\pi^2}(\overline{m}_u + \overline{m}_q)^2 a_s \lambda^2 \tau^{-1}, \quad (11)$$

Its value has been estimated from e^+e^- [28, 47] and τ -decay [29] data:

$$a_s \lambda^2 = -(0.07 \pm 0.03) \text{ GeV}^2. \quad (12)$$

• The instanton contribution

The inclusion of this contribution into the operator product expansion (OPE) is not clear and controversial [19–21]. In addition, an analogous contribution might lead to some contradiction to the OPE in the scalar channel [22]. Therefore, we shall consider the sum rule including the instanton contribution as an alternative approach. For our purpose, we parametrize this contribution as in [19, 20], where its corresponding contribution to the Laplace sum rule reads:

$$\mathcal{L}_5^P(\tau)|^{inst} = \frac{3}{8\pi^2}(m_u + m_q)^2 \tau^{-3} \rho_c^2 e^{-r_c} [K_0(r_c) + K_1(r_c)], \quad (13)$$

where K_i is the Bessel-Mac-Donald function; $r_c \equiv \rho_c^2/(2\tau)$ and $\rho_c = (1.89 \pm 0.11) \text{ GeV}^{-1}$ [48] is the instanton radius.

• Duality violation

Some eventual additional contribution from duality violation (DV) [49] could also be considered. However, as the LSR use the OPE in the Euclidian region where the DV effect is exponentially suppressed, one may safely neglect such contribution in the present analysis ⁷.

Table 1: Input parameters: the value of $\hat{\mu}_q$ has been obtained from the running masses evaluated at 2 GeV: $(\overline{m}_u + \overline{m}_d) = 7.9(6) \text{ MeV}$ [6, 50]. Some other predictions and related references can be found in [51]; ρ denotes the deviation on the estimate of the four-quark condensate from vacuum saturation. The error on $\Gamma_{K'}$ is a guessed conservative estimate. Most of the original errors have been enlarged for a conservative estimate of the errors.

Parameters	Values	Ref.
$\Lambda(n_f = 3)$	$(353 \pm 15) \text{ MeV}$	[30, 52]
\hat{m}_s	$(0.114 \pm 0.021) \text{ GeV}$	[6, 30, 50, 53]
$\hat{\mu}_d$	$(253 \pm 6) \text{ MeV}$	[50, 53]
$\kappa \equiv \langle \bar{s}s \rangle / \langle \bar{d}d \rangle$	$(0.74^{+0.34}_{-0.12})$	[6, 54]
$-a_s \lambda^2$	$(7 \pm 3) \times 10^{-2} \text{ GeV}^2$	[29, 47]
$\langle \alpha_s G^2 \rangle$	$(7.0 \pm 2.6) \times 10^{-2} \text{ GeV}^4$	[48]
M_0^2	$(0.8 \pm 0.2) \text{ GeV}^2$	[38–40]
$\rho \alpha_s \langle \bar{q}q \rangle^2$	$(5.8 \pm 1.8) \times 10^{-4} \text{ GeV}^6$	[27, 38, 41]
ρ_c	$(1.89 \pm 0.11) \text{ GeV}^{-1}$	[48]
$\Gamma_{\pi'}$	$(0.4 \pm 0.2) \text{ GeV}$	[51]
$\Gamma_{K'}$	$(0.25 \pm 0.05) \text{ GeV}$	[51]

⁷We thank the 2nd referee for this suggestion and for different provocative comments leading to the improvements of the final manuscript.

• The QCD input parameters

There are several estimates of the QCD input parameters in the current literature using different approaches and sometimes disagree each others. For a for self-consistency, we shall work in this paper with the input parameters given in Table 1 obtained using the same approach (Laplace or/and τ -decay-like sum rule) as the one used here and within the same criterion of stability (minimum, maximum or plateau in τ and t_c).

– \hat{m}_q and $\hat{\mu}_q$ are RGI invariant mass and condensates which are related to the corresponding running parameters as [2]:

$$\begin{aligned} \overline{m}_q(\tau) &= \hat{m}_q (-\beta_1 a_s)^{-2/\beta_1} (1 + \rho_m) \\ \langle \bar{q}q \rangle(\tau) &= -\hat{\mu}_q^3 (-\beta_1 a_s)^{2/\beta_1} / (1 + \rho_m) \\ \langle \bar{q}Gq \rangle(\tau) &= -M_0^2 \hat{\mu}_q^3 (-\beta_1 a_s)^{1/3\beta_1} / (1 + \rho_m), \end{aligned} \quad (14)$$

where $\beta_1 = -(1/2)(11 - 2n_f/3)$ is the first coefficient of the QCD β -function for n_f -flavours. ρ_m is the QCD correction which reads to N4LO accuracy for $n_f = 3$ [6, 55]:

$$\rho_m = 0.8951 a_s + 1.3715 a_s^2 + 0.1478 a_s^3, \quad (15)$$

where $a_s \equiv \alpha_s/\pi$ is the QCD running coupling. The value of \hat{m}_s quoted in Table 1 will serve as an initial value for the m_s corrections in the PT expression of the kaon correlator. It will be re-extracted by iteration in the estimate of m_s from the kaon sum rule where one obtains a convergence of the obvious iteration procedure after two iterations.

– The value of the μ_q RGI condensate used in Table 1 comes from the value $(\overline{m}_u + \overline{m}_d) = (7.9 \pm 0.6) \text{ MeV}$ evaluated at 2 GeV from [50] after the use of the GMOR relation:

$$2m_\pi^2 f_\pi^2 = -(m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle, \quad (16)$$

where $f_\pi = (92.23 \pm 0.14) \text{ MeV}$ [63].

– The value of the gluon condensate used here comes from recent charmonium sum rules. Since SVZ, several determinations of the gluon condensates exist in the literature [20, 28, 29, 32, 41, 47, 56–60]. The quoted error is about 2 times the original error for making this value compatible with the SVZ original value and charmonium analysis in [20] commented in [48] ⁸.

– We use the value of the four-quark condensate obtained from e^+e^- and VV+AA τ -decay [27–29, 41] data and from light baryons sum rules [38] where a deviation from the vacuum saturation by a factor $\rho \simeq (4.2 \pm 1.3)$ has been obtained if one evaluates $\langle \bar{d}d \rangle$ from μ_d given in Table 1 at M_τ where the four-quark condensate has been extracted (for a conservative result, we have multiplied the original error in [27] by a factor 2). Similar conclusions have been derived from FESR [56] and more recently from the VV-AA component of τ -decay data [60, 61]. We assume that a similar deviation holds in the pseudoscalar

⁸The sets of FESR in [56] tend give large values of the condensates which are in conflict with the ones from LSR and τ -like sum rules using similar e^+e^- data [27, 28, 41, 47] and previous charmonium analysis [3, 20, 32, 48]. They will not be considered here. However, as shall see explicitly later on, the effects of $\langle \alpha_s G^2 \rangle$ and of the tachyonic gluon mass used in this paper are relatively small in the present analysis.

channels. We shall see again later on that the error induced by this contribution on your estimate is relatively negligible.

– We use the value of the SU(3) breaking parameter $\kappa \equiv \langle \bar{s}s \rangle / \langle \bar{d}d \rangle$ from [54] which agrees with the ones obtained from from light baryons [38] and from kaon and scalar [7, 10, 12] sum rules recently reviewed in [6] but more accurate. For a conservative estimate we have enlarged the original error by a generous factor 4 and the upper value for recovering the central value $\kappa = 1.08$ from recent lattice calculations [62].

• LSR τ , t_c and μ stability criteria

The LSR is obtained within approximation both for the spectral side (when data are not available like here) and for the QCD side (as one has to truncate the PT series and the OPE at given orders). In the ideal case, where both sides of the LSR are perfectly described, one expects to find a large range of plateau stability (exact matching) at which one can extract the resonance parameters. It often happens that the minimal duality ansatz: “one resonance + QCD continuum from a threshold t_c ” description of the spectral function and/or the QCD approximation is rather crude. In this case, one can still extract an optimal information on the resonance parameters if the curves present a minimum, maximum or inflexion point versus the external LSR variable τ and the continuum threshold t_c as demonstrated in series of papers by Bell-Bertlmann [32] using the examples of harmonic oscillator and non-relativistic charmonium sum rules (see e.g. Figs. 49.6 and 49.7 pages 511-512 in Ref. [6]). At this minimum, maximum or inflexion point, one has a narrow sum rule window at which there is a balance between the QCD continuum and NP contributions and where the OPE still makes sense and the lowest resonances relatively dominates the spectral function. Analysis based on these criteria have been applied successfully in different applications of the sum rules (see e.g. Refs. [6, 7]). To these well-known τ and t_c stability criteria, we require a μ -stability of the results in order to limit the arbitrariness of the subtraction point μ often chosen ad hoc in the existing literature. Throughout this paper, we shall use the above criteria for extracting the optimal results from the analysis.

3. A Laplace sum rule estimate of the decay constant f_π

• The spectral function

We shall parametrize the spectral function as:

$$\frac{1}{\pi} \text{Im}\psi_5^\pi(t) \simeq \sum_{\pi, \pi'} 2f_P^2 m_P^4 \delta(t - m_P^2) + \text{QCD cont.} \theta(t - t_c), \quad (17)$$

where the higher states (π'', \dots) contributions are smeared by the “QCD continuum” coming from the discontinuity of the QCD diagrams and starting from a constant threshold t_c . f_P is the well-known decay constant :

$$\langle 0 | J_5^P(x) | P \rangle = \sqrt{2} m_P^2 f_P, \quad (18)$$

normalized here as: $f_\pi = (92.23 \pm 0.14) \text{ MeV}$ and $f_K \simeq (1.20 \pm 0.01) f_\pi$ [63]. We improve the $\pi' \equiv \pi(1300)$ contribution by

taking into account the finite width correction by replacing the delta function with a Breit-Wigner shape:

$$\pi \delta(t - m_{\pi'}^2) \rightarrow BW(t) = \frac{m_{\pi'} \Gamma_{\pi'}}{(t - m_{\pi'}^2)^2 + m_{\pi'}^2 \Gamma_{\pi'}^2}. \quad (19)$$

Defined in this way, the π' can be considered as an “effective resonance” parametrizing the higher state contributions not smeared by the QCD continuum and may take into account some possible interference between the $\pi(1300)$ and $\pi(1800)$ contributions.

• $f_{\pi'}$ from \mathcal{R}_5^π within the SVZ expansion at arbitrary μ ⁹

One expects from some chiral symmetry arguments that $f_{\pi'}$ behaves like m_π^2 . Therefore, one may expect that the π' will dominate over the pion contribution in the derivative of the Laplace sum rule:

$$-\frac{\partial}{\partial \tau} \mathcal{L}_5^\pi(\tau, \mu), \quad (20)$$

from which one can extract the decay constant $f_{\pi'}$ or the $\pi(1300)$ contribution to the spectral function. In order to eliminate the unknown value of the sum of light quark masses ($m_u + m_d$), it is convenient to work with the ratio of Finite Energy Laplace sum rules $\mathcal{R}_5^\pi(\tau, \mu)$ defined in Eq. (4). In so doing, we define the quantity:

$$r_\pi \equiv \frac{M_{\pi'}^4 f_{\pi'}^2}{m_\pi^4 f_\pi^2}, \quad (21)$$

which quantifies the relative weight between the π' and the pion contribution into the spectral function. It is easy to deduce the sum rule:

$$r_\pi = \frac{\mathcal{R}_5^{\pi|qcd} - m_\pi^2}{BW I_1 - \mathcal{R}_5^{\pi|qcd} BW I_0} e^{-m_\pi^2 \tau}. \quad (22)$$

$\mathcal{R}_5^{\pi|qcd}$ is the QCD expression of the FESR in Eq. (4) where we have parametrised the spectral function by a step function corresponding to the perturbative expression for massless quarks from the threshold t_c . $BW I_n$ is the Breit-Wigner integral:

$$BW I_n \equiv \frac{1}{\pi} \int_{9m_\pi^2}^{t_c} dt t^n e^{t\tau} BW(t) : \quad n = 0, 1, \quad (23)$$

where $BW(t)$ has been defined in Eq. (19).

– With the set of input parameters in Table 1, we show in Fig. 1a the τ -behaviour of r_π at a given value of $\mu = 1.55 \text{ GeV}$. We extract the optimal result at the value of $t_c = 2 \text{ GeV}^2$ and $\tau = (0.6 \pm 0.1) \text{ GeV}^{-2}$ where both a minimum in the change of t_c and an inflexion point in τ are obtained. One can notice that this value of t_c slightly higher than the $\pi(1.3)$ mass is inside the region of best stability between the spectral function and the QCD expression studied explicitly in [23]. For $\mu = 1.55 \text{ GeV}$,

⁹Here and in the following we shall denote by SVZ expansion the OPE without the instanton contribution.

at which we have an inflexion point for the central value, we deduce:

$$\begin{aligned} r_\pi^{\text{SVZ}} &= 4.43(14)_\Lambda(4)_{\lambda^2}(13)_{\bar{u}u}(31)_{G^2}(1)_{\bar{u}Gu}(20)_\rho \\ &\quad (161)_{\Gamma_\pi}(2)_{t_c}(10)_\tau \\ &= 4.43 \pm 1.67, \end{aligned} \quad (24)$$

where the dominant error comes from the experimental width of the $\pi(1300)$ which needs to be improved. The errors due to the QCD parameters are negligible despite the enlarged errors introduced for a conservative result.

– In Fig. 1b, we study the influence of the choice of μ varying in the range 1.4 to 1.8 GeV where a good duality between the QCD and spectral sides of the sum rules is obtained [23]. BPR has also noticed that the value of $t_c(s_0 = \mu^2$ in their notation) is below the $\pi(1.8)$ mass and there is a complex interference between the $\pi(1.3)$ and $\pi(1.8)$ indicating the complexity of the pseudoscalar spectral function. Therefore, for quantifying the $\pi(1.8)$ contribution, we study in Fig. 1c, its effect by using one of the models proposed by BPR with the mixing parameter $\zeta = 0.234 + i 0.1$ which is the one which reproduces the best fit to the experimental curves which observe the $\pi(1.8)$ in hadronic interactions [23]. We also compare our results with the one in Ref. [31] by taking the central value: $f_{\pi(1.8)} \approx 1.36$ MeV where no interference with the $\pi(1.3)$ has been assumed. We notice from the analysis in Fig. 1c that, in both cases, the $\pi(1.8)$ effect is negligible.

– Our final result corresponds to the mean of different determinations in Fig. 1b, where the dashed coloured region corresponds to the final error ± 1.56 from the most precise determination \oplus the systematics 0.17 corresponding to the distance of the mean to this precise determination. added quadratically:

$$r_\pi^{\text{SVZ}} = 4.45 \pm 1.56 \pm 0.17_{\text{sys}}, \quad (25)$$

• Convergence of the QCD series

Here, we study the convergence of the PT series in the more general case where μ is arbitrary and not necessarily correlated to the value of τ . The particular case $\mu = \tau^{-1/2}$ often used in the literature will be also discussed in the next paragraph.

– We study in Fig 2a, the convergence of the PT QCD series at the value of the subtraction scale $\mu = 1.55$ GeV where a μ stability is obtained (inflexion point in Fig. 1b) for the ratio $\mathcal{R}_5^\pi(\tau, \mu)$ used for the estimate of r_π (lower family of curves). One can notice that for $\tau \approx 0.6$ GeV⁻² where a τ inflexion is obtained (Fig. 1a), the α_s , α_s^2 , α_s^3 and α_s^4 effects are respectively -7.5 , -9.4 , -9.0 , and -4.5% of the preceding PT series: LO, up to NLO, up to N2LO up to N3LO and up to N4LO or equivalently, the PT series behaves as: $1 - 0.07 - 0.08 - 0.08 - 0.05$. The convergence of the PT series is slow but each corrections to r_π are reasonably small. However, one can notice that the convergence of the PT series is much better here than in the case (often used in the literature) $\mu = \tau^{-1/2}$ where the τ stability is obtained at larger value of $\tau = 1.9$ GeV⁻² (Fig. 5a) as we shall discuss later on.

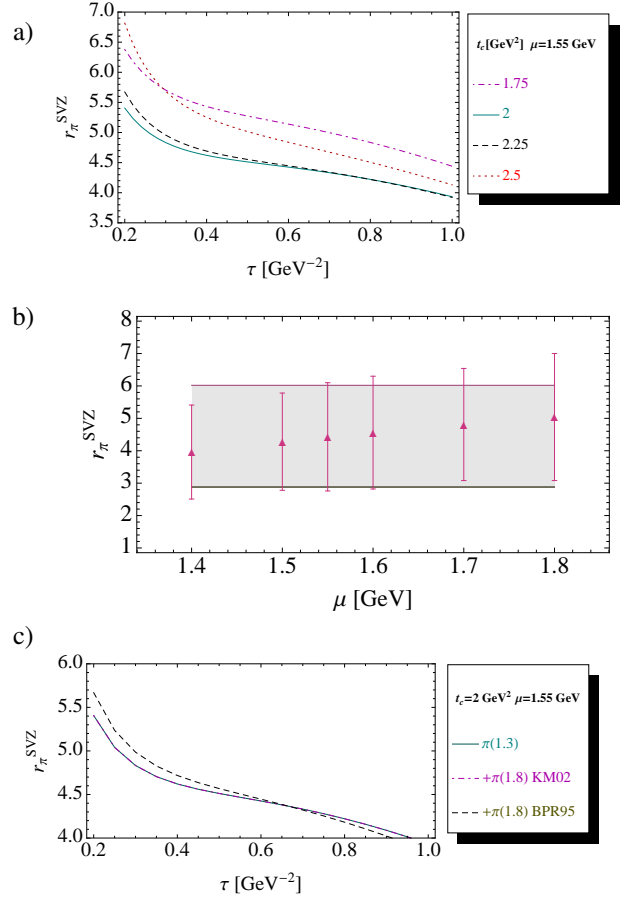


Figure 1: a) τ -behaviour of r_π for $\mu = 1.55$ GeV and for different values of t_c within the SVZ expansion. ; b) μ -behaviour of the optimal value of r_π deduced from a). The coloured region corresponds to the mean value where the error comes from the most precise determination \oplus the same systematics; c) Comparison of the effects of $\pi(1.8)$ for two different models of the spectral function where one can remark a complete coincidence of the $\pi(1.3)$ curve and $\pi(1.3) + \pi(1.8)$ from KM02.

– We show in Fig 2b, the convergence of the power corrections for r_π (lower family of curves). We see that the $d = 2, 4, 6$ dimension operator effects are $-1.4, -4.2$ and -1.4% of the preceding sum of contributions or equivalently, the NP series normalized to the PT series behaves as: $1 - 0.018 - 0.065 - 0.046$ indicating a slow convergence of the OPE but relatively small corrections.

• Tachyonic gluon mass and large order PT-terms to r_π

The tachyonic gluon mass decreases the value of r_π by about 0.1 which is relatively negligible. The smallness of this contribution is consistent with the small contribution of the estimated N5LO terms using a geometric growth of the PT series. Using the duality between the long PT series and the short PT series $\oplus 1/Q^2$ correction [45], the inclusion of the $1/Q^2$ into the OPE mimics the contributions of the non-calculated higher order in the PT series which are therefore expected to be small numbers. Here and in the following the PT series is truncated at N4LO (order α_s^4) and the sum of α_s^n corrections for $n \geq 5$ is approximated by the tachyonic gluon mass contribution.

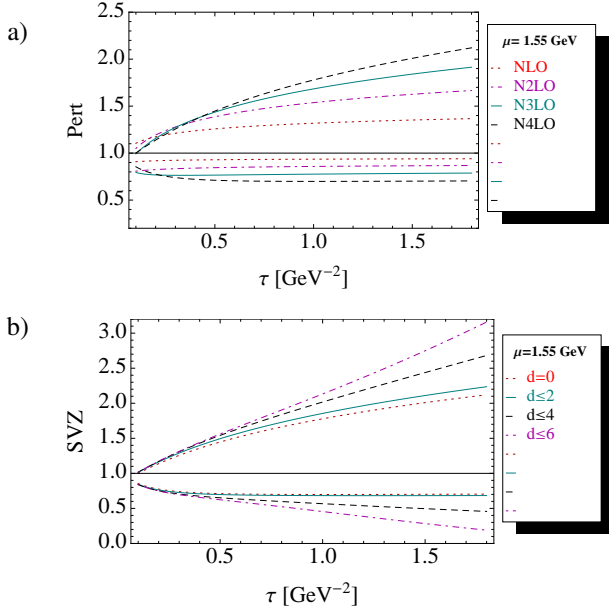


Figure 2: a) τ -behaviour of the PT series of $\sqrt{L_5^\pi(\tau, \mu)}$ (upper group of curves) and of $R_5^\pi(\tau, \mu)$ (lower group of curves) appropriately normalised to 1 for $\tau = 0$ and using $\mu = 1.55$ GeV. ; b) the same as a) but for the power corrections within the SVZ expansion.

• r_π from instanton sum rule at arbitrary μ

We include the instanton contribution into the OPE using the expression given in Eq. (13). The variations of r_π versus τ and t_c for different values of μ are similar to the one in Fig. 1a. The optimal result is obtained for $\tau \simeq (0.9 \pm 0.1)$ GeV⁻² (inflexion point) and $t_c \simeq 2.25$ GeV² (minimum in t_c). Comparing the behaviour of the curves using the SVZ and the SVZ \oplus instanton expansion in Fig. 3a, one can notice that the instanton contribution has shifted the inflexion point at slightly higher τ -values. Normalized to the PT contributions, the sum of the SVZ term to $R_5^\pi(\tau, \mu)$ at $\tau \simeq 0.9$ GeV⁻² is -33% of the PT contribution while the one of the instanton is about +77% (see Fig. 4). At $\mu = 1.55$ GeV and for $\tau = (0.9 \pm 0.1)$ GeV⁻², we obtain after an iteration procedure by using the final value of $\langle \bar{d}d \rangle$ condensate obtained in Eq. (42) for the $d = 4$ condensate contribution¹⁰:

$$\begin{aligned} r_\pi^{inst} &= 2.11(1)_\Lambda(3)_{\lambda^2}(10)_{\bar{u}u}(10)_{G^2}(1)_{\bar{u}Gu}(17)_\rho(1)_{\rho_c} \\ &\quad (50)_{\Gamma_\pi}(3)_{t_c}(5)_\tau \\ &= 2.11 \pm 0.57. \end{aligned} \quad (26)$$

We study in Fig. 3b the μ behaviour of the optimal results in the range $\mu = 1.4$ to 1.8 GeV like in the previous analysis. Then, we deduce the mean value:

$$r_\pi^{inst} = 2.06 \pm 0.55 \pm 0.20_{syst}, \quad (27)$$

where the first error comes from the most precise determination, while the systematic error is the distance of the mean from it.

¹⁰The value of the four-quark condensate extracted from the V and V+A channels quoted in Table 1 is expected [21, 30] to be weakly affected by instanton in the OPE.

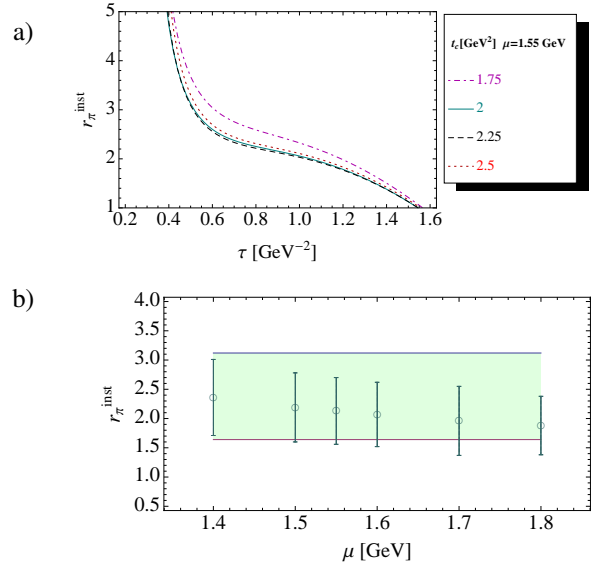


Figure 3: a) τ -behaviour of r_π for $\mu = 1.55$ GeV and for different values of t_c from the instanton sum rule. b) μ -behaviour of the optimal value of r_π deduced from a) and sources of errors analogue to the ones in Fig. 1.

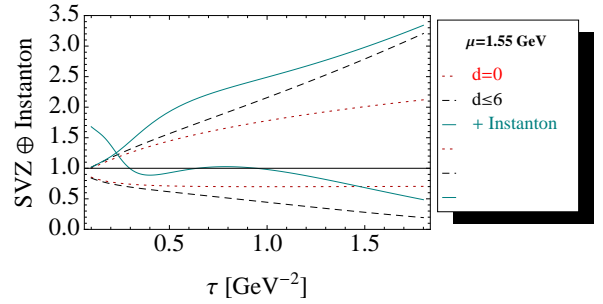


Figure 4: τ -behaviour of the NP corrections including the instanton contribution to $\sqrt{L_5^\pi(\tau, \mu)}$ (upper group of curves) and of $R_5^\pi(\tau, \mu)$ (lower group of curves) appropriately normalised to 1 for $\tau = 0$ and using $\mu = 1.55$ GeV.

• r_π from LSR at $\mu = \tau^{-1/2}$

We complete the analysis in the case where the subtraction constant μ is equal to the sum rule variable $1/\sqrt{\tau}$. This case is interesting as it does not possess the $\text{Log}^n \mu^2 \tau$ terms appearing in the PT series which have large coefficients and which are now absorbed into the running of $\alpha_s(\tau)$ from the renormalization group equation. This case has been largely used in the literature (for reviews see, e.g.: [5–8]). The analysis is very similar to the previous case. In Fig. 5, we show the τ -behaviour of the results in the case of the SVZ expansion and SVZ \oplus instanton contribution where in both cases a minimum in t_c is obtained. We obtain for $\tau = (2 \pm 0.1)$ GeV⁻² and $t_c = 2$ GeV².

$$\begin{aligned} r_\pi^{SVZ} &= 4.36(120)_\Lambda(12)_{\lambda^2}(81)_{\bar{u}u}(67)_{G^2}(1)_{\bar{u}Gu}(169)_\rho \\ &\quad (56)_{\Gamma_\pi}(2)_{t_c}(4)_\tau \\ &= 4.36 \pm 2.39, \end{aligned} \quad (28)$$

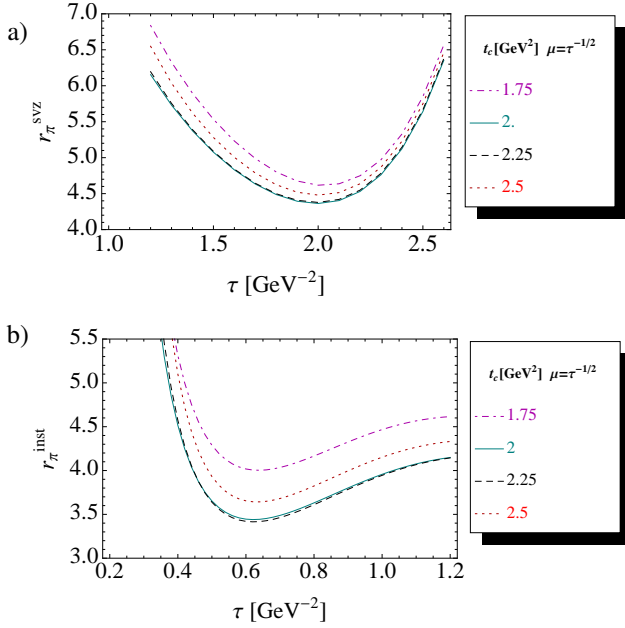


Figure 5: a) τ -behaviour of r_π for the case $\mu = \tau^{-1/2}$ in the case of the SVZ expansion; b) the same as in a) but for SVZ \oplus instanton contribution.

For the instanton case, we obtain for $\tau = (0.6 \pm 0.1) \text{ GeV}^{-2}$ and $t_c = 2.25 \text{ GeV}^2$:

$$\begin{aligned} r_\pi^{\text{inst}} &= 3.40(8)_\Lambda(1)_{\lambda^2}(7)_{\bar{u}u}(14)_{G^2}(1)_{\bar{u}Gu}(15)_p(22)_{p_c} \\ &\quad (104)_{\Gamma_\pi}(2)_{t_c}(4)_\tau \\ &= 3.40 \pm 1.09 . \end{aligned} \quad (29)$$

• Final result and comparison with some existing predictions

One can remark a nice agreement within the errors between the different results in Eq. (25) with Eq. (28) and Eq. (27) with Eq. (29). However, the estimates from the LSR with $\mu = \tau^{-1/2}$ are obtained at much larger values of τ and are sensitive to the NP contributions rendering the estimate less accurate. Taking the mean of the previous estimates, we deduce our final results:

$$\begin{aligned} r_\pi^{\text{SVZ}} &= 4.42 \pm 1.56 \implies \frac{f_{\pi'}}{f_\pi} = (2.42 \pm 0.43) 10^{-2}, \\ r_\pi^{\text{inst}} &= 2.36 \pm 0.74 \implies \frac{f_{\pi'}}{f_\pi} = (1.77 \pm 0.28) 10^{-2}, \end{aligned} \quad (30)$$

where we have separated the determinations from the SVZ and SVZ \oplus instanton sum rules. The errors come from the most precise estimate to which we have added a systematic from the distance of the mean to it. In Fig. 6, we compare the above two results, with the existing ones in the current literature: NPT83 [12], SN02 [6], BPR95 [23], KM02 [31] for the quantity:

$$L_\pi(\tau) \equiv r_\pi BW I_0 , \quad (31)$$

involved in the Laplace sum rule estimate of $(m_u + m_d)$ which we shall discuss in the next sections. Here BWI_0 defined in Eq. (23) is the integrated spectral function entering into the lowest moment Laplace sum rule $\mathcal{L}_5^\pi(\tau)$. For this comparison, we have used:

– $r_\pi = (9.5 \pm 2.5)$ and consistently the NWA for the results in NPT83 and SN02 from [6] (see also [16]).

– For KM02 [31], we add coherently the $\pi(1300)$ and $\pi(1800)$ contributions which may be an overestimate as they may have a destructive interference like in BPR95 [23]. We use the decay constants $f_\pi(1300) = (2.2 \pm 0.57) \text{ MeV}$ and $f_\pi(1800) = (1.36 \pm 0.21) \text{ MeV}$ obtained in [31] and consistently a Breit-Wigner parametrization of the spectral function.

– For BPR95, we add, into their parametrization, the error due to the $\pi(1300)$ width which is not included in their original work and we use the mixing parameter $\zeta = 0.0234 + i 0.1$ between the $\pi(1300)$ and $\pi(1800)$ which reproduces the best fit to the experimental curves which observe the $\pi(1.8)$ in hadronic interactions.

– The CHPT parametrization from BPR95 [23] without any resonance is also given in Fig. 6.

The results obtained in [23] and [31] are model-dependent as they depend on the way of treating the $\pi(1800)$ contribution into the spectral function. One can see explicitly in Fig. 1c that the $\pi(1800)$ contribution to r_π is negligible rendering the result in this paper less model-dependent thanks to the exponential weight of the LSR which safely suppresses its contribution.

From the previous comparison, we notice that the prediction from the SVZ expansion has a better agreement (within the errors) with the predictions shown in Fig. 6, while the one including the instanton tends to underestimate the $\pi(1300)$ contribution to the spectral function. One can also notice that the NWA used in NPT83 [12] and SN02 [6] used there tends to give larger values which presumably indicates that the NWA is not sufficient for a good description of the pseudoscalar spectral function.

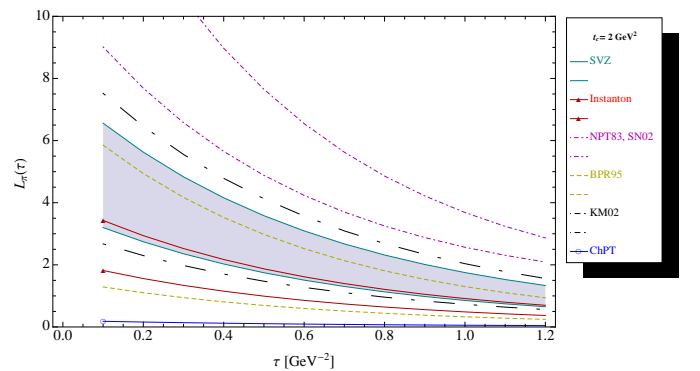


Figure 6: Comparison of some other determinations of r_π for a given value of $t_c = 5 \text{ GeV}^2$ which corresponds to the optimal value of $(m_u + m_d)$. The blue continuous line with a circle is the ChPT prediction without a resonance. The results of NPT83 [12] and SN02 [6] are within a narrow width approximation. The errors due to the experimental width of the $\pi(1300)$ have been introduced in the result of BPR95 [23].

4. Estimate of $(m_u + m_d)$ within the SVZ expansion

• The LSR for arbitrary μ

We find convenient to extract the RGI scale independent mass defined in Eq. (14):

$$\hat{m}_{ud} \equiv \frac{1}{2}(\hat{m}_u + \hat{m}_d) \quad (32)$$

from the Laplace sum rule $\mathcal{L}_5^\pi(\mu, \tau)$ in Eq. (3). The QCD expression of $\mathcal{L}_5^\pi(\mu, \tau)$ is given in Eq. (5). We shall use into the spectral function, parametrized as in Eq. (17), the value of r_π obtained in Eq. (30) and we do not transfer the QCD continuum contribution to the QCD side of the sum rule. In this way, we obtain a much better τ -stability but we have an initial value of \hat{m}_{ud} for quantifying the QCD continuum contribution. Therefore, we use an obvious iteration procedure by replacing successively the initial input value of \hat{m}_{ud} with the obtained value and so on. The procedure converges rapidly after 2 iterations. We show, in Fig. 7a, the τ -dependence of \hat{m}_{ud} for different values of t_c and for a given value of the subtraction point $\mu = 1.55$ GeV, where the optimal value of r_π has been obtained. One can find from this figure that one obtains a τ -stability for $\tau \simeq (0.7 \pm 0.1)$ GeV⁻². A minimum in t_c is also obtained for $t_c \simeq (2 \sim 2.25)$ GeV² consistent with the one in the determination of r_π . Using the values of the parameters in Table 1, we extract the optimal value of the sum of the RGI u, d quark masses for $\mu=1.55$ GeV and at the extrema (stability region) of the curve:

$$\begin{aligned} \hat{m}_{ud}^{svz} &= 4.56(11)_\Lambda(6)_{\lambda^2}(1)_{\bar{u}u}(10)_{G^2(0)\bar{u}Gu}(7)_\rho \\ &\quad (10)_{\Gamma_\pi}(27)_{r_\pi}(0)_{\pi(1.8)}(0)_\tau(2)_{t_c} \text{ MeV} , \\ &= (4.56 \pm 0.32) \text{ MeV} , \end{aligned} \quad (33)$$

where the errors due to the localisation of the τ and t_c stability region are negligible like also the $\pi(1.8)$ contribution using its coupling from [31]. We study the μ -dependence of the result in Fig. 7b and deduce the mean value:

$$\langle \hat{m}_{ud}^{svz} \rangle = (4.59 \pm 0.31 \pm 0.06_{\text{syst}}) \text{ MeV} , \quad (34)$$

which corresponds to the one of the slight μ inflexion point obtained around $(1.55 - 1.60)$ GeV. The first error is the one from the most precise measurement. The second one is a systematics coming from the distance of the mean to its central value.

• \hat{m}_{ud} from the Laplace sum rule at $\mu = \tau^{-1/2}$

As mentioned previously, this sum rule has been widely used in the current literature for extracting m_{ud} . We shall use it here as another method for determining m_{ud} . The analysis is similar to the one for arbitrary μ . We show the τ -dependence for different t_c in Fig. 8. One can see that unlike the case of arbitrary μ , there is no τ -stability here. Therefore, we shall not consider this approach in this analysis.

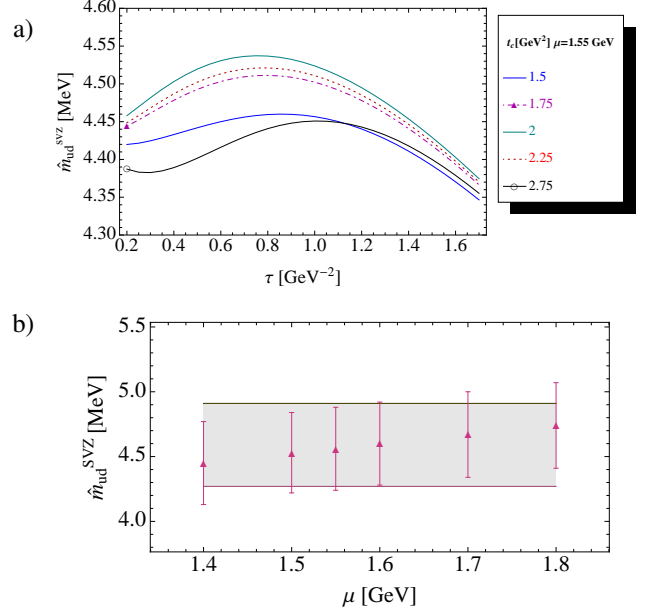


Figure 7: τ - and t_c -dependence of \hat{m}_{ud} from the Laplace sum in Eq. (3) at the subtraction scale $\mu = 1.55$ GeV. The filled coloured region corresponds to mean value where the errors come from the most precise determination \oplus systematics.

• Convergence of the QCD series

Like in the case of r_π , we study the convergence of the PT series and of the OPE.

– We shall study the different contributions of the truncated PT series to $\sqrt{\mathcal{L}_5^\pi(\mu, \tau)}$ for $\mu = 1.55$ GeV where a μ stability is obtained (slight inflexion point in Fig. 1b). The relative strengths of each truncated contributions are given in Fig 2a (upper family of curves). One can deduce that for $\tau \approx 0.7$ GeV⁻² where the τ -stability is obtained (Fig. 7), the α_s , α_s^2 , α_s^3 and α_s^4 effects are respectively +29, +13, +6 and +3% of the preceding PT series: LO, NLO, N2LO and N3LO, which is equivalent to: $1 + 0.29 + 0.17 + 0.09 + 0.05$ when normalized to the LO PT contribution. It indicates a good convergence of the PT series.

– We show in Fig 2b, the convergence of the power corrections for $\sqrt{\mathcal{L}_5^\pi(\mu, \tau)}$ for $\mu = 1.55$ GeV (upper family of curves). We see that the $d = 2, 4, 6$ contributions are +3.8, +4.8 and +2.9% of the preceding sum of contributions (PT , $PT \oplus d = 2$, $PT \oplus d = 2 + 4$) or equivalently: $1 + 0.04 + 0.05 + 0.03$ when normalized to the PT series indicating a slow convergence but relatively small corrections.

• Tachyonic gluon mass and large order PT-terms to \hat{m}_{ud}

If we do not include the tachyonic gluon mass contribution into the SVZ expansion, the value of \hat{m}_{ud} obtained in Eq. (33) would increase by 0.15 MeV which is relatively negligible confirming again the good convergence of the PT series if one evokes a duality between the tachyonic gluon mass and the not yet calculated higher order PT corrections [45]. Within this duality argument, one can estimate the contribution of the large order non calculated PT terms (sum of the higher order α_s^n : $n \geq 5$) by the one of the tachyonic gluon mass.

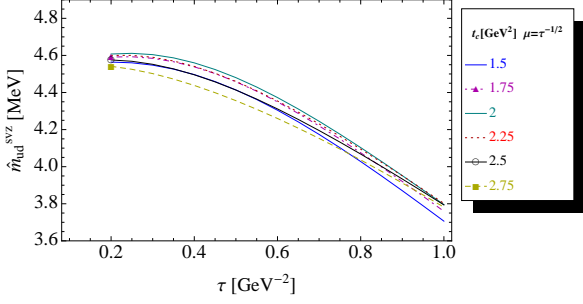


Figure 8: τ - and t_c -dependence of \hat{m}_{ud} from the Laplace sum in Eq. (3) at the subtraction scale $\mu = \tau^{-1/2}$.

• Final estimate of \hat{m}_{ud} within the SVZ expansion

We consider, as a *final estimate* of \hat{m}_{ud} within the SVZ expansion, the mean value in Eq. (34) which is:

$$\langle \hat{m}_{ud}^{SVZ} \rangle = (4.59 \pm 0.32) \text{ MeV}. \quad (35)$$

5. m_{ud} from the instanton Laplace sum rules

For optimizing the instanton contribution, we work at the same subtraction point $\mu = (1.4 - 1.8) \text{ GeV}$ where r_π^{inst} has been obtained. We shall use the value of r_π extracted in Eq. (30). We repeat the previous analysis by taking into account the instanton contribution. Its contribution to $\sqrt{\mathcal{L}_5^\pi(\mu, \tau)}$ compared to the OPE up to $d=6$ condensates is shown in Fig. 4 (upper family of curves) for $\mu = 1.55 \text{ GeV}$ where the estimate of r_π has been also optimized. For $\tau \approx (0.4 \sim 0.5) \text{ GeV}^{-2}$ where the sum rule is optimized (Fig. 9a and Fig. 9b for $\mu = \tau^{-1/2}$ and $\mu = 1.55 \text{ GeV}$), the instanton contribution is about +36% (resp 15%) of the perturbative $\oplus d \leq 6$ condensates for the sum rule subtracted at $\mu = \tau^{-1/2}$ (resp. $\mu = 1.55 \text{ GeV}$). t_c -stability is reached for $t_c \simeq (2 \sim 2.25) \text{ GeV}^2$. We deduce respectively from the sum rule subtracted at $\mu = \tau^{-1/2}$ and $\mu = 1.55 \text{ GeV}$:

$$\begin{aligned} \hat{m}_{ud}^{inst}|_{1.55} &= 2.81(4)_\Lambda(2)_\Lambda^2(0)_{\bar{u}u}(1)_{G^2(0)}(0)_{\bar{u}Gu}(1)_\rho \\ &\quad (7)_{\rho_c}(6)_{\Gamma_\pi}(13)_{r_\pi}(1)_\tau(0)_{t_c} \text{ MeV}, \\ &= (2.81 \pm 0.17) \text{ MeV}, \\ \hat{m}_{ud}^{inst}|_{\tau^{-1/2}} &= 2.76(6)_\Lambda(2)_\Lambda^2(0)_{\bar{u}u}(2)_{G^2(0)}(0)_{\bar{u}Gu}(1)_\rho \\ &\quad (6)_{\rho_c}(6)_{\Gamma_\pi}(19)_{r_\pi}(2)_\tau(0)_{t_c} \text{ MeV}, \\ &= (2.76 \pm 0.22) \text{ MeV}. \end{aligned} \quad (36)$$

We show in Fig. 9c the μ behaviour of the different determinations from which we deduce the mean value with the conservative error:

$$\langle \hat{m}_{ud}^{inst} \rangle \simeq (2.81 \pm 0.16 \pm 0.10_{syst}) \text{ MeV}, \quad (37)$$

which we consider as a determination of \hat{m}_{ud} from the SVZ \oplus instanton sum rules.

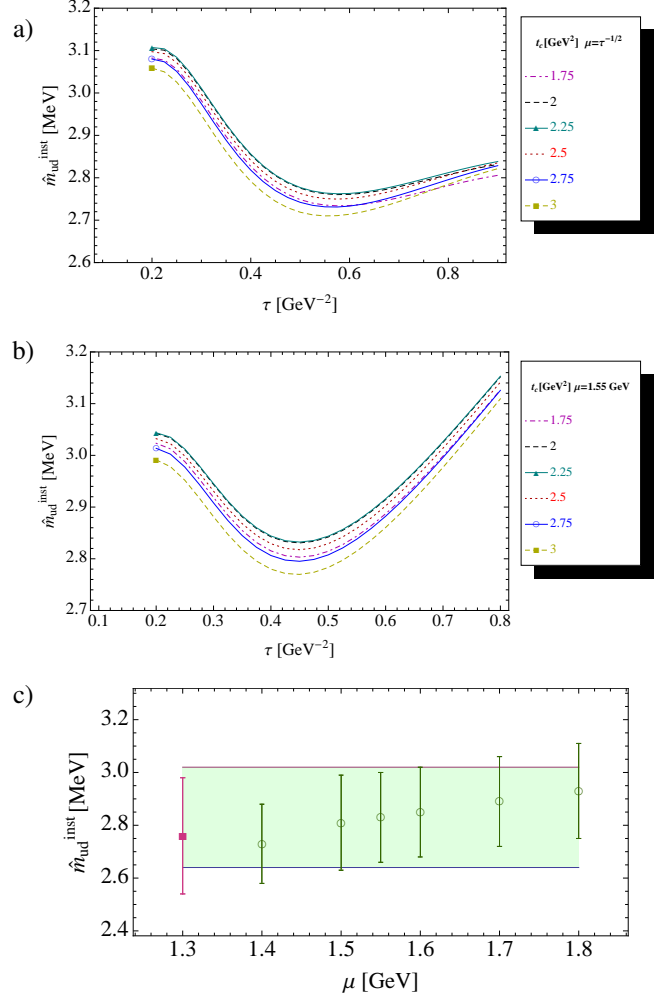


Figure 9: a) τ -behaviour of m_{ud} in the case $\mu = \tau^{-1/2}$; b) the same as in a) but in the case $\mu = 1.55 \text{ GeV}$; c) μ -behaviour of \hat{m}_{ud} obtained from LSR: the red box is the value from $\mu = \tau^{-1/2}$ in b). Same meaning of coloured region as in Fig. 7b.

6. \hat{m}_{ud} and $\bar{m}_{ud}(2)$ from Laplace sum rules

We consider, as a *final estimate* of the RGI mass \hat{m}_{ud} , the results obtained in Eqs. (35) and (37) from which we deduce the running masses at order α_s^4 evaluated at 2 GeV in units of MeV:

$$\bar{m}_{ud}^{SVZ} = 3.95 \pm 0.28, \quad \bar{m}_{ud}^{inst} = 2.42 \pm 0.16. \quad (38)$$

We have not taken the mean value of the two results taking into account the controversial contribution of the instanton into the pseudoscalar sum rule [19–22]. The value of the sum \bar{m}_{ud} within the SVZ expansion agrees within the errors with the average: $\bar{m}_{ud} = (5.0 \pm 0.9) \text{ MeV}$, quoted in [6] and coincide with the one $\bar{m}_{ud} = (3.95 \pm 0.30) \text{ MeV}$, deduced by combining the value of the average \bar{m}_s from different phenomenological sources and the ChPT mass ratio m_s/m_{ud} [50]. We consider the previous results as improvements of our previous determinations in [6] from the Laplace pseudoscalar sum rules and some other sum rules determinations in this channel compiled in PDG13 [51]. The previous results can also be compared with the recent lattice average [64]:

$$\bar{m}_{ud}^{latt} = (3.6 \pm 0.2) \text{ MeV} \text{ [resp. } (3.4 \pm 0.1) \text{ MeV]}, \quad (39)$$

obtained using $n_f = 2$ [resp. $n_f = 2 + 1$] dynamical fermions where there is a good agreement within the errors with the SVZ value but not with the instanton one which is lower. Using the mean of the range of different results quoted in PDG13 [51] for the ratio:

$$\frac{m_u}{m_d} = 0.50(3), \quad (40)$$

which (a priori) does not favour the solution $m_u = 0$ advocated in connection with the strong CP-problem (see e.g [65]), one can deduce the value of the u and d running quark masses at 2 GeV in units of MeV:

$$\begin{aligned} \bar{m}_u^{svz} &= 2.64 \pm 0.28, & \bar{m}_u^{inst} &= 1.61 \pm 0.14 \\ \bar{m}_d^{svz} &= 5.27 \pm 0.49, & \bar{m}_d^{inst} &= 3.23 \pm 0.29. \end{aligned} \quad (41)$$

Using the GMOR relation in Eq. (16), we can deduce the value of the running light quark condensate: $\langle \bar{u}u \rangle \simeq \langle \bar{d}d \rangle$ at 2 GeV in units of MeV^3 :

$$-\langle \bar{d}d \rangle^{svz} = (276 \pm 7)^3, \quad -\langle \bar{d}d \rangle^{inst} = (325 \pm 7)^3, \quad (42)$$

and to the spontaneous mass in units of MeV defined in Eq. (14):

$$\mu_d^{svz} = 253 \pm 6, \quad \mu_d^{inst} = 298 \pm 7, \quad (43)$$

where the SVZ result is in perfect agreement with the one from [50] used in Table 1. The results are summarized in Table 3.

7. Laplace sum rule estimate of $f_{K'}$

Using the same method as in the case of the π' , we shall estimate the $K' \equiv K(1460)$ decay constant through:

$$r_K \equiv \frac{M_{K'}^4 f_{K'}^2}{m_K^4 f_K^2}. \quad (44)$$

• Analysis within the SVZ expansion for arbitrary μ

– We show the τ - and t_c -behaviours of r_K in Fig. 10a for a given value of the subtraction point $\mu = 2.15$ GeV, where the τ -maximum is obtained at 0.7 GeV^{-2} and an almost plateau from $\tau \simeq (0.6 \sim 0.9) \text{ GeV}^{-2}$ for $t_c \simeq (3.05 \pm 0.10) \text{ GeV}^2$. At this scale, one can inspect using curves similar to Fig. 2a (lower families of curves) that the PT corrections to $\mathcal{R}_S^K(\tau, \mu)$ are small: the $\alpha_s, \alpha_s^2, \alpha_s^3$ and α_s^4 effects are respectively -6% , -6.4% , -6.8% and -6.1% of the preceding PT series including: LO, NLO, N2LO and N3LO contributions which is equivalent to: $1 - 0.06 - 0.06 - 0.06 - 0.05$ when normalized to the LO perturbative series. The NP corrections similar to the ones in Fig. 2b remain reasonably small: the $d = 2, 4, 6$ dimension operators contributions are -1.3% , -10.5% and -4.3% of the preceding sum of contributions ($PT, PT \oplus d = 2, PT \oplus d = 2 + 4$) which is equivalent to: $1 - 0.01 - 0.10 - 0.04$ when normalized to the PT contributions.

– In Fig. 10b, we show the μ -behaviour of the central values of the optimal results obtained from Fig. 10a. One can notice a

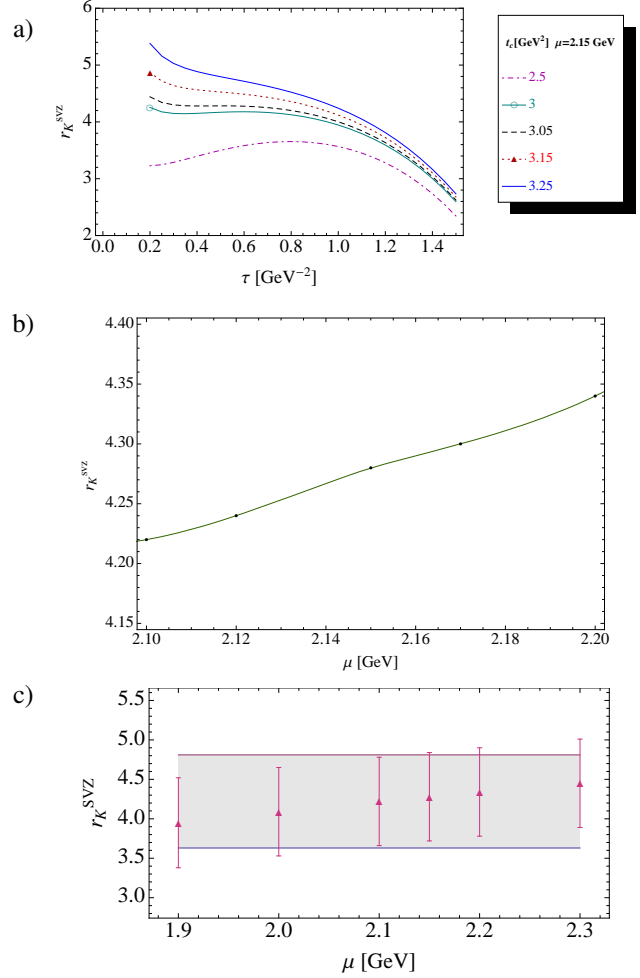


Figure 10: **a)** τ -behaviour of r_K for a given value $\mu = 2.15$ GeV of subtraction point and for different values of t_c ; **b)** μ -behaviour of the optimal central values of r_K deduced from **a)**; **c)** μ -behaviour and mean value of r_K from LSR: same meaning of coloured region as in Fig. 1b.

slight inflexion point like in the case of the pion. At this point $\mu = 2.15$ GeV, we obtain:

$$\begin{aligned} r_K^{svz} &= 4.28(6)_{\Lambda}(6)_{\lambda^2}(11)_{\bar{u}u}(31)_{G^2}(1)_{\bar{u}Gu}(25)_{\rho} \\ &\quad (1)_{m_s}(-24)_{+11}(30)_{\Gamma_K}(10)_{t_c} \\ &= 4.28 \pm 0.56. \end{aligned} \quad (45)$$

In Fig. 10c, we study the effects of μ by moving it from 1.9 to 2.3 GeV around the inflexion point. The average of these results leads to the final estimate:

$$r_K^{svz} = 4.22 \pm 0.54 \pm 0.23_{sys} \Rightarrow \frac{f_{K'}}{f_K} = (23.5 \pm 1.6) 10^{-2}, \quad (46)$$

where one can remark from Eq. 30 that $r_\pi \approx r_K$ as expected from chiral symmetry arguments.

• Analysis within the SVZ expansion for $\mu = \tau^{-1/2}$

We show the result of the analysis in Fig. 11 where there is no τ stability. Therefore, we shall not consider the result of this sum rule in the following.

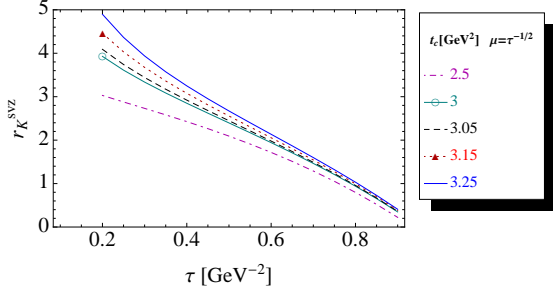


Figure 11: τ -behaviour of r_K for $\mu = \tau^{-1/2}$ and for different values of t_c .

• r_K from instanton sum rules

The analysis of the sum rule for arbitrary μ does not lead to a conclusive result. The one for $\mu = \tau^{-1/2}$ is given in Fig. 12. Like in the case of the pion, we shall use the value of the $\langle \bar{d}d \rangle$ condensate obtained in Eq. (42) for the $d = 4$ condensate contribution and the value in Table 1 for the four-quark condensate extracted from the V and V+A channels which is weakly affected by instanton effects [21, 30]. We deduce for $t_c = (3.0 - 3.05) \text{ GeV}^2$:

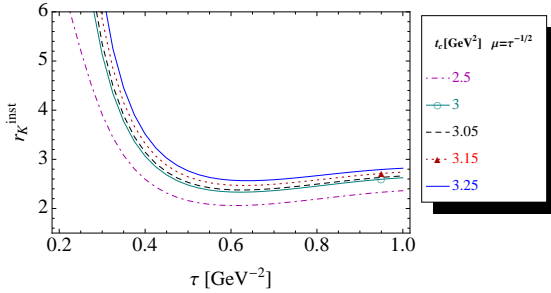


Figure 12: τ -behaviour of r_K for a given value $\mu = \tau^{-1/2}$ of the subtraction point and for different values of t_c from the instanton sum rule

$$\begin{aligned} r_K^{inst} &= 2.36(17)_{SVZ}(16)_{\rho_c}(12)_{\Gamma_K}(2)_{t_c}(0)_\tau \\ &= 2.36 \pm 0.26 \Rightarrow \frac{f_{K'}}{f_K} = (17.6 \pm 1.0)10^{-2}. \end{aligned} \quad (47)$$

where the index SVZ means that the corresponding error is the quadratic sum of the ones due to the PT contributions and to the NP terms within the SVZ expansion defined in Section 2:

$$(17)_{SVZ} \equiv (6)_\Lambda(0)_{\lambda^2}(4)_{\bar{u}u}(10)_{G^2}(2)_{\bar{u}Gu}(9)_\rho(1)_{m_s}(6)_K. \quad (48)$$

• Comparison with some other predictions

We compare in Fig. 13, our results from Eqs. (46) and (47) for:

$$L_K(\tau) \equiv r_K BW I_0, \quad (49)$$

with the existing ones in the current literature (NPT83 [12], SN02 [6], KM02 [31] and DPS98 [24]). The results of NPT83 [12] and SN02 [6] are obtained within a narrow width approximation. The ones of KM02 [31] and DPS98 [24] include finite

width correction. There are fair agreement between different determinations with the exception of the one from [24] which is relatively high (central value shown in Fig. 13). This high value may be either due to the coherent sum and equal coupling of the $K(1460)$ and $K(1800)$ contribution assumed in the amplitude or due to an overall normalization factor¹¹. We also see in Fig. 13 that the instanton sum rule estimate is relatively small compared with the one from the sum rule within the SVZ expansion and with some other determinations. As we shall see later, this low value of the $K(1460)$ contribution will imply a smaller value of m_s from the instanton sum rule estimate.

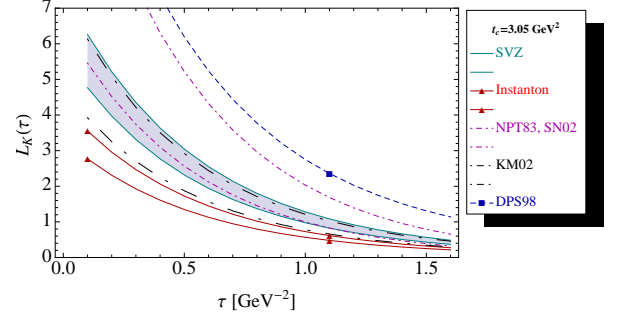


Figure 13: Comparison of our determination of r_K from SVZ and instanton sum rules with the ones in the current literature: NPT83 [12], SN02 [6], KM02 [31] and DPS98 [24]). We use $t_c = 3.05 \text{ GeV}^2$.

8. Laplace sum rule estimate of \hat{m}_{us} and \bar{m}_{us}

Defining:

$$m_{us} = (m_u + m_s), \quad (50)$$

we now turn to the estimate of the RGI \hat{m}_{us} and running \bar{m}_{us} sum of masses.

• \hat{m}_{us} within the SVZ expansion for arbitrary μ

We show in Fig. 14a the τ -behaviour of \hat{m}_{us} for a given value $\mu = 2.15 \text{ GeV}$ of the subtraction point and for different values of t_c where we have used the value of r_K in Eq. (46). The largest range of τ -stability of about $(0.6 - 0.9) \text{ GeV}^{-2}$ is reached at $t_c \simeq (3.6 \pm 0.1) \text{ GeV}^2$. However, one can notice that the value of t_c corresponding to the best stability for \hat{m}_{us} differs slightly with the one $t_c \simeq (3.05 \pm 0.10) \text{ GeV}^2$ for r_K . We consider this systematics by enlarging the range of t_c to $t_c \simeq (3.6 \pm 0.4) \text{ GeV}^2$. Using the initial value of m_{us} in Table 1 and after two obvious iterations, we obtain for $\mu = 2.15 \text{ GeV}$:

$$\begin{aligned} \hat{m}_{us}^{SVZ} &= 118.0(30)_\Lambda(16)_{\lambda^2}(6)_{\bar{u}u}(23)_{G^2}(1)_{\bar{u}Gu}(13)_\rho \\ &\quad (5)_K(6)_{\Gamma_K}(40)_{r_K}(4)_{t_c}(0)_\tau \text{ MeV} \\ &= (118.0 \pm 6.0) \text{ MeV}, \end{aligned} \quad (51)$$

¹¹Notice that instead of [24] in the kaon channel, a destructive interference has been assumed by [23] in the pion channel, with which agrees our estimate in the pion channel.

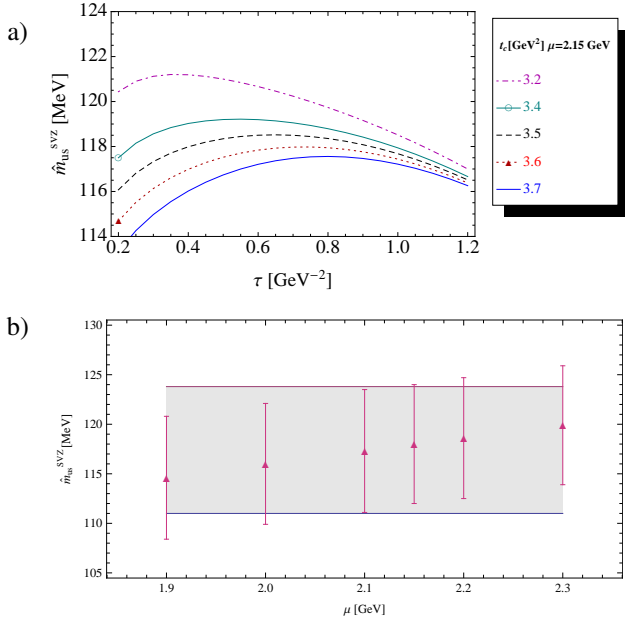


Figure 14: **a)** τ -behaviour of \hat{m}_{us} for a given value $\mu = 2.15$ GeV of subtraction point and for different values of t_c ; **b)** μ -behaviour of the optimal value of \hat{m}_{us} deduced from a); same meaning of coloured region as in Fig. 7b.

where the main error comes from the K' -meson contribution. We show in Fig. 14b the μ -behaviour of the optimal value of \hat{m}_{us} from Fig. 14a from which we deduce the mean value:

$$\langle \hat{m}_{us}^{SVZ} \rangle = (117.4 \pm 5.9 \pm 2.5_{\text{sys}}) \text{ MeV}. \quad (52)$$

- \hat{m}_{us} within the SVZ expansion for $\mu = \tau^{-1/2}$

We redo the previous analysis but for $\mu = \tau^{-1/2}$. Unfortunately, we have no stability like in the case of r_K .

- \hat{m}_{us} from the instanton sum rule at arbitrary μ

We show in Fig. 15 the τ -behaviour of \hat{m}_{us} from the instanton sum rule at different values of t_c and for a given value $\mu = 2.15$ GeV. We take the optimal value at the τ -minimum of about (0.45 ± 0.10) GeV⁻² where we notice that the effect of t_c in the range (3.6 ± 0.4) GeV² is relatively small. We obtain:

$$\begin{aligned} \hat{m}_{us}^{inst} &= 79.3(16)_\Lambda(7)_{\lambda^2}(2)_{\bar{u}u}(5)_{G^2}(0)_{\bar{u}Gu}(1)_\rho \\ &\quad (24)_{\rho_c}(2)_K(5)_{\Gamma_K}(20)_{r_K}(15)_{t_c}(12)_\tau \text{ MeV} \\ &= (79.3 \pm 4.1) \text{ MeV}. \end{aligned} \quad (53)$$

We study the μ -dependence of the results in Fig. 15 from which we deduce the mean value:

$$\hat{m}_{us}^{inst} = (78.9 \pm 4.1 \pm 0.4_{\text{sys}}) \text{ MeV}. \quad (54)$$

- \hat{m}_{us} from the instanton sum rule at $\mu = \tau^{-1/2}$

The analysis is shown in Fig. 16. Our optimal results correspond to $\tau = (0.5 \sim 0.9)$ GeV⁻² and $t_c = (3.6 \pm 0.4)$ GeV². We deduce

$$\begin{aligned} \hat{m}_{us}^{inst} &= 70.65(132)_\Lambda(65)_{\lambda^2}(14)_{\bar{u}u}(70)_{G^2}(4)_{\bar{u}Gu}(27)_\rho \\ &\quad (157)_{\rho_c}(142)_{r_K}(-15)_{\kappa}(62)_{\Gamma_K}(60)_{t_c}(30)_\tau \text{ MeV} \\ &= (70.7 \pm 2.8) \text{ MeV}. \end{aligned} \quad (55)$$

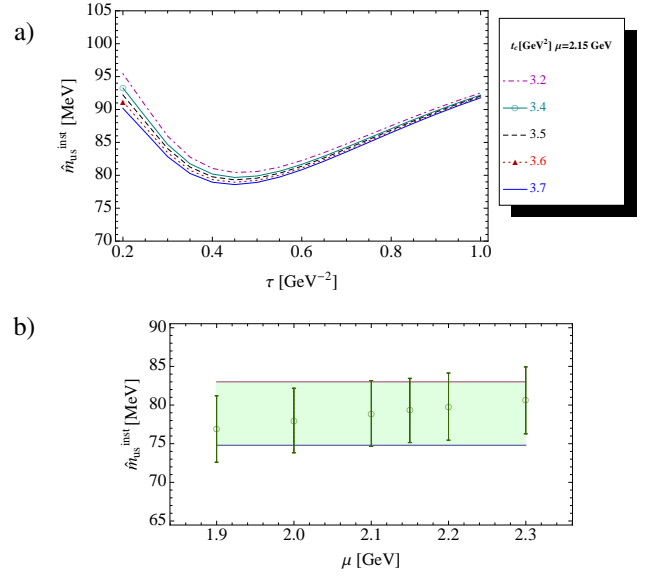


Figure 15: **a)** τ -behaviour of \hat{m}_{us} from the instanton sum rule for a given value $\mu = 2.15$ GeV of the subtraction point and for different values of t_c ; **b)** μ -behaviour of the optimal results: same meaning of coloured region as in Fig. 7b.

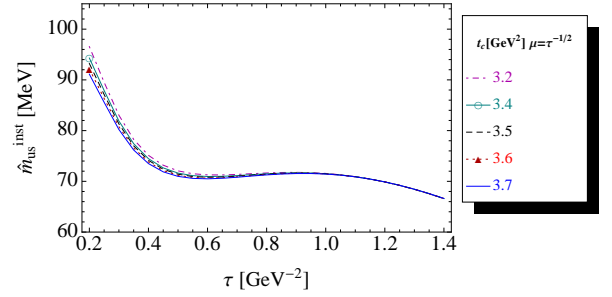


Figure 16: τ -behaviour of \hat{m}_{us} from the instanton sum rule for $\mu = \tau^{-1/2}$ and for different values of t_c .

- Final value of \hat{m}_{us} and \bar{m}_{us}

Our final results are from Eq. (52) for the SVZ expansion and from the combination of the one from Eqs. (54) and (55) from the instanton sum rule. One obtains in units of MeV:

$$\hat{m}_{us}^{SVZ} = 117.4 \pm 6.4, \quad \hat{m}_{us}^{inst} = 73.3 \pm 3.9, \quad (56)$$

where the errors on \hat{m}_{us}^{inst} is the quadratic sum of the one from the most precise determination and the systematics estimated from the distance of the mean to the central value of this precise determination. The corresponding running masses evaluated at 2 GeV are:

$$\bar{m}_{us}^{SVZ} = 101.1 \pm 5.5, \quad \bar{m}_{us}^{inst} = 63.1 \pm 3.4. \quad (57)$$

Using as input the values of \bar{m}_u given in Eq. (41), one can deduce:

$$\bar{m}_s^{SVZ} = 98.5 \pm 5.5, \quad \bar{m}_s^{inst} = 61.5 \pm 3.4. \quad (58)$$

Combining this result with the value of \bar{m}_{ud} in Eq. (38), one predicts the scale-independent mass ratios:

$$\left(\frac{m_s}{m_{ud}} \right)^{SVZ} = 24.9 \pm 2.3, \quad \left(\frac{m_s}{m_{ud}} \right)^{inst} = 25.4 \pm 2.2, \quad (59)$$

and:

$$\left(\frac{m_s}{m_d}\right)^{\text{svz}} = 18.7 \pm 2.0, \quad \left(\frac{m_s}{m_d}\right)^{\text{inst}} = 19.0 \pm 2.0. \quad (60)$$

These results are summarized in Table 3 where one can remark that unlike the absolute values of the light quark masses, their ratios are almost unaffected by the presence of instanton in the OPE. These results agree within the errors with the previous determinations in [6, 50]. One can also compare these results with the recent lattice average for $n_f = 2 \oplus 1$ flavours [64]:

$$\bar{m}_s^{\text{latt}} = (93.8 \pm 2.4) \text{ MeV}, \quad \left(\frac{m_s}{m_{ud}}\right)^{\text{latt}} = 27.44 \pm 0.44, \quad (61)$$

where a good agreement with the ratio and with the value of \bar{m}_s^{svz} is observed while the one of \bar{m}_s^{inst} is too low.

9. Lower bounds on \hat{m}_{uq} and \bar{m}_{uq} from Laplace sum rule

Lower bounds on quark masses have been first derived in [1, 9] and improved later on in [66] using a finite number of Q^2 -derivatives of the two-point function. Using the second derivative of the two-point function defined in Eq. (1) which is super-convergent:

$$\psi_5''(Q^2) = \int_{m_\pi^2}^{\infty} dt \frac{2}{(t+Q^2)^3} \text{Im}\psi_5(t), \quad (62)$$

retaining the pion pole and using the positivity of the spectral function, one can derive the (linear) lower bound:

$$(\bar{m}_u + \bar{m}_d)(Q^2) \geq 4\pi \sqrt{\frac{2}{3}} \frac{m_\pi^2 f_\pi}{Q^2}, \quad (63)$$

at lowest order of PT QCD. In [7], the α_s^3 corrections to the result to order α_s^2 of [66] have been included leading to the (improved) lower linear bounds evaluated at $Q=2$ GeV:

$$\begin{aligned} \bar{m}_{ud} &\equiv \frac{1}{2}(\bar{m}_u + \bar{m}_d) \geq (3.0 \pm 0.5) \text{ MeV}, \\ \bar{m}_{us} &\equiv (\bar{m}_u + \bar{m}_s) \geq (82.7 \pm 13.3) \text{ MeV}, \end{aligned} \quad (64)$$

In [18], order α_s^4 corrections have been added for improving the previous bound on \bar{m}_{us} and lead to a result consistent with the one in Eq. (64). However, there is no other arguments for fixing the value of Q^2 obtained in Eq. (63) apart the convergence of PT series at which the bound is evaluated. As the $1/Q^2$ fall off of the bound is faster than the Q^2 behaviour of the running mass in Eq. (14) predicted by the RGE, the bound becomes relatively weak at larger Q^2 -values. In the present work, we shall use the Laplace sum rule $\mathcal{L}_5^P(\tau, \mu)$ (linear constraint) defined in Eq. (3) together with the optimization procedure used in previous sections for extracting an “optimal lower bound” on the sum of light quark RGI scale-independent masses ($\hat{m}_u + \hat{m}_q$) which we shall translate later on to bounds on the running quark masses ($\bar{m}_u + \bar{m}_q$). In so doing, we shall use the positivity of the “QCD continuum contribution” by taking ($t_c \rightarrow \infty$) in Eq. (3) and we shall only consider the meson pole contributions to the spectral function. We shall also include (for the first time for the Laplace sum rules) the α_s^4 PT corrections for deriving these bounds.

• Bounds from Laplace sum rules at $\mu = \tau^{-1/2}$

We study the lower bounds obtained from $\mathcal{L}_5^P(\tau, \mu)$ sum rule within the SVZ expansion (Fig. 17a) and the one where the instanton contribution is added into the OPE (Fig. 17b). We have only retained the pion contribution into the spectral function. Similar curves are obtained in the s -quark channel (Fig. 18). Among the different bounds associated to τ shown in these figures, the most stringent one (hereafter denoted “optimal bound”) on the quark invariant masses which are scale independent will be extracted at the maximum or / and at the τ -stability region where one has both a good control of the OPE and an optimal contribution of the resonances to the spectral function. For $\mu = \tau^{-1/2}$, we obtain¹² in units of MeV:

$$\begin{aligned} \hat{m}_{ud}^{\text{svz}} &\geq 2.79(14)_\Lambda(4)_{\lambda^2}(4)_{\bar{u}u}(8)_{G^2}(0)_{\bar{u}Gu}(9)_\rho \\ &\geq (2.79 \pm 0.19), \\ \hat{m}_{us}^{\text{svz}} &\geq 74.6(30)_\Lambda(10)_{\lambda^2}(7)_{\bar{u}u}(15)_{G^2}(0)_{\bar{u}Gu}(4)_\rho(7)_\kappa \\ &\geq (74.6 \pm 3.7), \end{aligned} \quad (65)$$

and:

$$\hat{m}_{ud}^{\text{inst}} \geq 2.47(16)_{\text{svz}}(0)_{\rho_c}, \quad \hat{m}_{us}^{\text{inst}} \geq 62.5(28)_{\text{svz}}(1)_{\rho_c}. \quad (66)$$

At the scale $\tau \approx 1 \text{ GeV}^{-2}$ where these optimal bounds are extracted (maximum in τ), the NLO, N2LO, N3LO and N4LO PT QCD corrections to these bounds normalized to the LO contributions are respectively:

$$\text{PT} = \text{LO} (1 + 0.32 + 0.22 + 0.14 + 0.10), \quad (67)$$

indicating a reasonable convergence of the PT series. A duality between the PT series and the tachyonic mass contribution [45] leads to an estimate of about 0.04 of the uncalculated large order terms contributions. At this scale the d=4 and 6 condensate contributions are respectively 9% and 8% of the total PT contributions while the instanton contribution is 28%. One can translate the previous bounds on the RGI masses into the ones for the running masses evaluated at 2 GeV in units of MeV:

$$\begin{aligned} \bar{m}_{ud}^{\text{svz}} &\geq 2.41 \pm 0.15, \quad \bar{m}_{us}^{\text{svz}} \geq 64.3 \pm 3.1, \\ \bar{m}_{ud}^{\text{inst}} &\geq 2.13 \pm 0.14, \quad \bar{m}_{us}^{\text{inst}} \geq 53.8 \pm 2.4. \end{aligned} \quad (68)$$

Using the value of the ratio m_u/m_d in Eq. (40), one can deduce from the bound on \bar{m}_{ud} in units of MeV:

$$\begin{aligned} \bar{m}_u^{\text{svz}} &\geq 1.61 \pm 0.10, \quad \bar{m}_u^{\text{inst}} \geq 1.42 \pm 0.13, \\ \bar{m}_d^{\text{svz}} &\geq 3.21 \pm 0.20, \quad \bar{m}_d^{\text{inst}} \geq 2.84 \pm 0.25. \end{aligned} \quad (69)$$

Using the GMOR relation in Eq. (16), one can translate the previous lower bounds on \bar{m}_{ud} into upper bounds for the running quark condensate evaluated at 2 GeV:

$$-\langle \bar{u}u \rangle^{\text{svz}} \leq (325 \pm 7)^3, \quad -\langle \bar{u}u \rangle^{\text{inst}} \leq (339 \pm 7)^3, \quad (70)$$

and for the spontaneous mass in units of MeV defined in Eq. (14):

$$\mu_u^{\text{svz}} \leq 298 \pm 6, \quad \mu_u^{\text{inst}} \leq 311 \pm 6. \quad (71)$$

¹²Within the present approximation, the sum rules with an arbitrary μ do not present a τ -stability and will not be useful here.

Using the value of \bar{m}_u in Eq. (41), one can deduce, from the bound on m_{us} , the ones of running masses at 2 GeV, in units of MeV:

$$\bar{m}_s^{svz} \geq 61.5 \pm 3.1, \quad \bar{m}_s^{inst} \geq 52.3 \pm 3.4. \quad (72)$$

Though weaker than the ones in Eq. (64), these “optimal bounds” are interesting as previously discussed. The results are summarized in Table 3.

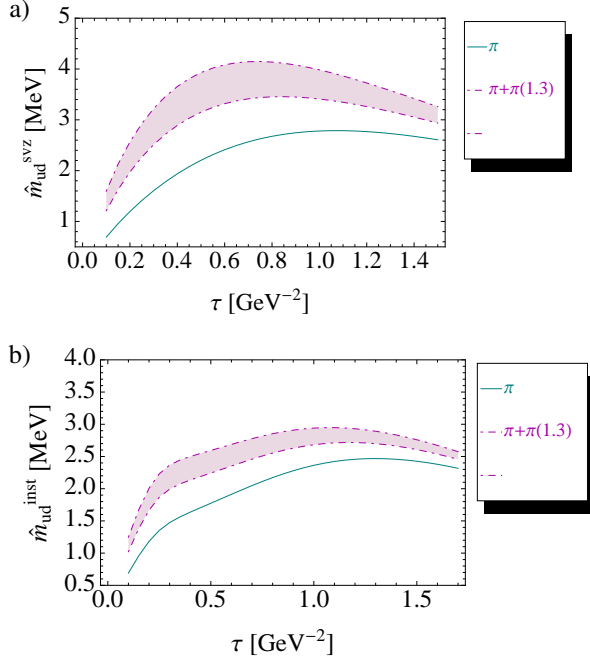


Figure 17: **a)** τ -behaviour of the lower bound of \hat{m}_{ud} from the sum rule within the SVZ expansion for $\mu = \tau^{-1/2}$, continuous line: pion contribution only, shaded region: inclusion of the $\pi(1.3)$; **b)** the same as in a) but for the instanton sum rule.

• $\pi(1.3)$ and $K(1.46)$ effects on the previous bounds

If one includes the contribution of the $\pi(1.3)$ [resp. $K(1.46)$] in the pion [respectively kaon] spectral function, one can improve the previous bounds. The effect of the $\pi(1.3)$ and $K(1.46)$ are shown respectively in Fig. 17 and Fig. 18. The optimal bounds become (in units of MeV):

$$\begin{aligned} \hat{m}_{ud}^{svz}|_{\pi(1.3)} &\geq 3.81(14)_{\Lambda(6)\lambda^2(4)\bar{u}u(8)G^2(0)\bar{u}Gu(11)_\rho} \\ &\quad (8)_{\Gamma_\pi(38)_{r_\pi}} \\ &\geq (3.81 \pm 0.41), \\ \hat{m}_{us}^{svz}|_{K(1.46)} &\geq 97.8(31)_{\Lambda(14)\lambda^2(5)\bar{u}u(13)G^2(1)\bar{u}Gu(11)_\rho} \\ &\quad (\bar{c}_{+2}^6)_\kappa(1)_{\Gamma_K(36)_{r_K}} \\ &\geq (97.8 \pm 5.2), \end{aligned} \quad (73)$$

and:

$$\begin{aligned} \hat{m}_{ud}^{inst} &\geq 2.84(12)_{\pi(1.3)(17)_{svz}(0)_{\rho_c}}, \\ \hat{m}_{us}^{inst} &\geq 70.3(10)_{K(1.46)(34)_{svz}(4)_{\rho_c}}, \end{aligned} \quad (74)$$

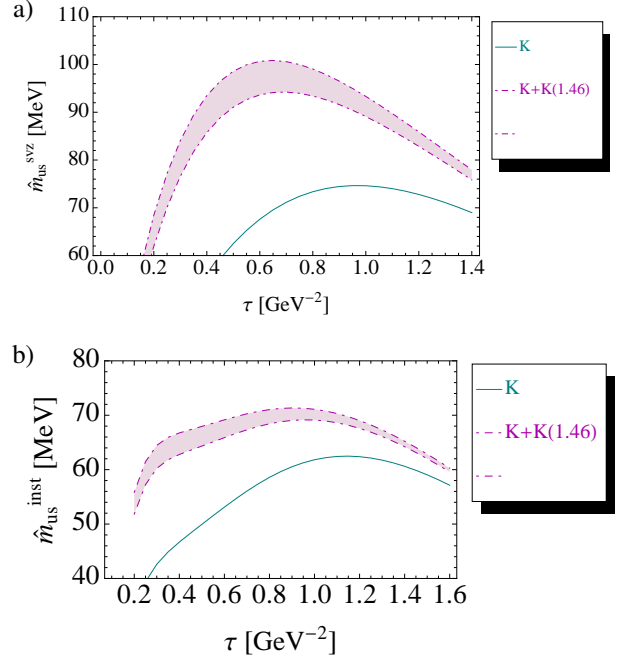


Figure 18: **a)** τ -behaviour of the lower bound of \hat{m}_{us} from the sum rule within the SVZ expansion for $\mu = \tau^{-1/2}$, continuous line: kaon contribution only, shaded region: inclusion of the $K(1.46)$; **b)** the same as in a) but for the instanton sum rule.

which can be translated into the ones for the running masses evaluated at 2 GeV in units of MeV:

$$\begin{aligned} \bar{m}_{ud}^{svz} &\geq 3.28 \pm 0.35, & \bar{m}_{us}^{svz} &\geq 84.2 \pm 4.5, \\ \bar{m}_{ud}^{inst} &\geq 2.45 \pm 0.18, & \bar{m}_{us}^{inst} &\geq 60.5 \pm 3.1. \end{aligned} \quad (75)$$

and:

$$\begin{aligned} \bar{m}_u^{svz} &\geq 2.19 \pm 0.27, & \bar{m}_u^{inst} &\geq 1.64 \pm 0.16, \\ \bar{m}_d^{svz} &\geq 4.37 \pm 0.54, & \bar{m}_d^{inst} &\geq 3.27 \pm 0.31. \end{aligned} \quad (76)$$

Like previously, the lower bounds on \bar{m}_{ud} can be translated into upper bounds for the running quark condensate evaluated at 2 GeV:

$$-\langle \bar{u}u \rangle^{svz} \leq (294 \pm 11)^3, \quad -\langle \bar{u}u \rangle^{inst} \leq (324 \pm 9)^3, \quad (77)$$

and for the spontaneous mass in units of MeV defined in Eq. (14):

$$\mu_u^{svz} \leq 267 \pm 10, \quad \mu_u^{inst} \leq 297 \pm 9. \quad (78)$$

Using the previous value of \bar{m}_u in Eq. (41), one can deduce from Eq (75), the bounds, on the running masses evaluated at 2 GeV, in units of MeV:

$$\bar{m}_s^{svz} \geq 81.6 \pm 4.5, \quad \bar{m}_s^{inst} \geq 58.9 \pm 3.1. \quad (79)$$

The “optimal bounds” obtained in Eq. (75) for the quark running masses from the linear sum rules based on the SVZ expansion and including the $\pi(1300)$ (resp. $K(1460)$) are slightly stronger than the ones given in Eq. (64) obtained from finite number of derivatives. One may consider the present bounds as alternatives to the ones in the existing literature [1, 7, 9, 18, 66].

Summary and conclusions

We have re-estimated the $\pi(1300)$ and $K(1460)$ decay constants using pseudoscalar Laplace sum rules which we have compared with some existing ones in the literature. We have used these results for improving the determinations of (m_u+m_q) : $q \equiv d, s$ from these channels. Our results obtained from the set of parameters in Table 2 are summarized in Table 3. The

Table 2: Values of the set external parameters (μ, τ, t_c) obtained at the stability points corresponding to the optimal values of r_P and m_q . μ is in GeV, τ in GeV^{-2} and t_c in GeV^2 . The indices SVZ and inst correspond to the SVZ and SVZ \oplus instanton expansions.

Observables	μ	τ	$\tau = \mu^{-2}$	t_c
<i>Pion channel</i>				
r_π^{SVZ}	1.4 – 1.8	0.5 – 0.7	1.7 – 2.1	2.0 – 2.25
r_π^{inst}	–	0.8 – 1.0	0.5 – 0.7	–
m_{ud}^{SVZ}	–	0.7 – 0.9	unstable	–
m_{ud}^{inst}	–	0.4 – 0.5	0.4 – 0.7	–
<i>Kaon channel</i>				
r_K^{SVZ}	1.9 – 2.3	0.6 – 0.9	unstable	3.05 – 3.25
r_K^{inst}	–	unstable	0.5 – 0.7	–
m_{us}^{SVZ}	–	0.5 – 0.9	unstable	3.2 – 4.0
m_{us}^{inst}	–	0.35 – 0.55	0.5 – 0.9	–

novel features in the present analysis are:

- In addition to the usual sum rule evaluated at $\mu = \tau^{-1/2}$ where τ is the Laplace sum rule variable, we have used an arbitrary subtraction point μ in the range 1.4-1.8 GeV [23] where the best duality between the QCD and experimental sides of the pion sum rules is obtained. Its most precise values have been fixed from a μ -stability criterion [inflexion point (Figs. 1b and 14b) or an (almost) stable plateau (Figs. 10 and 15) or an extremum (Fig. 3) depending on the sum rule used] and is given in Table 2. The sets of (τ, t_c) , values parameters which optimize the duality between the experimental and QCD sides of each sum rule from stability criteria are summarized in Table 2 and come from Figs. 1a, 3a, 5, 7 to 10a, 11, 12a, 14a, 15a and 16. Their values may differ for each form of the sum rules analyzed due to the different reorganization of the QCD series and the relative weight of different resonances in the spectral integral for each form of sum rules. In most cases analyzed in this paper, the Laplace sum rules within the SVZ expansion and for arbitrary value of μ show a large region of plateau stability, while the $\tau^{-1/2} = \mu$ and the one within the SVZ expansion \oplus instanton show only extremal points which in some cases are reached for large values of τ and induce some additional errors not present in the one within the SVZ expansion and for arbitrary value of μ .
- Unlike the well-known case of ρ meson channel, where the continuum threshold t_c can be interpreted to be approximately the value of the 1st radial excitation ρ' meson mass [6, 7], the situation for the pseudoscalar mesons are quite different due, presumably, to the Goldstone nature of the pion, where the 1st radial excitation $\pi(1300)$ strongly dominates in the estimate of

Table 3: Summary of the main results of this work. For deriving the values and bounds of $\bar{m}_{u,d}$, we have used $m_u/m_d = 0.50 \pm 0.03$ deduced from the compilation of PDG13 [51]. The estimated value of \bar{m}_u has been used for an estimate and for giving a bound on \bar{m}_s . The running masses \bar{m}_q evaluated at 2 GeV are in units of MeV. The bounds for the quark masses are lower bounds while the ones for the $\langle \bar{u}u \rangle$ quark condensate are upper bounds.

Estimates	SVZ	SVZ \oplus instanton	Eq.
$f_{\pi'}/f_\pi$	$(2.42 \pm 0.43)10^{-2}$	$(1.77 \pm 0.28)10^{-2}$	30
$f_{K'}/f_K$	$(23.5 \pm 1.6)10^{-2}$	$(17.6 \pm 1.0)10^{-2}$	46, 47
\bar{m}_{ud}	3.95 ± 0.28	2.42 ± 0.16	38
\bar{m}_u	2.64 ± 0.28	1.61 ± 0.14	41
\bar{m}_d	5.27 ± 0.49	3.23 ± 0.29	41
$\langle \bar{u}u \rangle$	$-(276 \pm 7)^3$	$-(325 \pm 7)^3$	70
\bar{m}_{us}	101.1 ± 5.5	63.1 ± 3.4	57
\bar{m}_s	98.5 ± 5.5	61.5 ± 3.4	58
m_s/m_{ud}	24.9 ± 2.3	25.7 ± 2.2	59
m_s/m_d	18.7 ± 2.0	19.0 ± 2.0	60

Bounds	SVZ		SVZ \oplus Instanton		Eq.
	π	$\pi \oplus \pi(1.3)$	π	$\pi \oplus \pi(1.3)$	
\bar{m}_{ud}	2.41 ± 0.15	3.28 ± 0.35	2.13 ± 0.14	2.45 ± 0.18	68, 75
\bar{m}_u	1.61 ± 0.10	2.19 ± 0.27	1.45 ± 0.09	1.64 ± 0.16	69, 76
\bar{m}_d	3.21 ± 0.20	4.37 ± 0.54	2.84 ± 0.19	3.27 ± 0.31	69, 76
$\langle \bar{u}u \rangle$	$-(325 \pm 7)^3$	$-(294 \pm 11)^3$	$-(339 \pm 7)^3$	$-(324 \pm 9)^3$	70, 77
\bar{m}_{us}	K	$K \oplus K(1.46)$	K	$K \oplus K(1.46)$	
	64.3 ± 3.1	84.2 ± 4.5	53.8 ± 2.4	60.5 ± 3.1	68, 75
	61.5 ± 3.1	81.6 ± 4.5	52.1 ± 2.4	58.9 ± 3.1	72, 79

r_π while they act almost equally in the determination of m_{ud} . Also, in this pseudoscalar channel, there can be a possible negative interference between the $\pi(1300)$ and the second radial excitation $\pi(1800)$ as emphasized by [23], which is not the case of the ρ meson channel. For this reason, it may be misleading to give a physical interpretation of t_c here as its value should be affected by the relative weight between the contributions of the two different resonances π and $\pi(1300)$ and their eventual interferences in the spectral integral as well as the reorganization of the QCD perturbative and non-perturbative series in each sum rules. Here, the alone solid constraint that one can impose is that t_c should be above the mass of the lowest resonances $\pi(1.3)$ and $K(1.46)$ analyzed where t_c is 2 (resp 3) GeV^2 for the pion (resp. kaon) channel.

- The improved model-independent extraction of the experimentally unknown contribution of the $\pi(1300)$ and $K(1460)$ into the spectral function and the inclusion of finite width corrections. These results agree with the models presented in Fig. 6 and Fig. 13. One can also notice that the contributions of the 2nd radial excitation $\pi(1.8)$ and $K(1.8)$ are negligible as shown explicitly in Fig. 1c.
- An inclusion of the tachyonic gluon mass into the SVZ ex-

pansion showing that its effect is relatively small (it decreases r_π and r_K by 0.1 and \hat{m}_{ud} (resp. \hat{m}_{us}) by 0.13 (resp. 3) MeV) which is reassuring. This negligible effect together with the picture of duality [45] between the tachyonic gluon mass contribution and the sum of uncalculated higher order terms of the QCD PT series indicates that these large order effects are negligible at the scale where we extract the optimal results which can be explicitly checked by an estimate of the N5LO contribution based on the geometric growth of the PT series.

- An explicit study of the Laplace sum rule including instanton contribution which we have considered as an alternative determination of $(m_u + m_d)$ despite the controversial role of the instanton into the pseudoscalar sum rule. However, the relative small contribution of the $\pi(1300)$ and $K(1460)$ to the spectral function from this analysis (see the comparison in Fig. 6 and Fig. 13) does not (a priori) favour this contribution which consequently induces relative small values of the quark masses compared to the one using the standard SVZ expansion (see Table 3 and [6, 51]) and recent lattice calculations [51, 64].

- One may consider our results as improvements of the existing analytical determinations of $m_{ud} \equiv (m_u + m_d)/2$: $q \equiv d, s$ from the pseudoscalar Laplace sum rules since the first analysis of [1]. We have not taken the mean value of the two different determinations from SVZ without instanton and from SVZ \oplus instanton due to the controversial instanton role into the pseudoscalar sum rules. The results using the SVZ expansion without the instanton contribution can be compared with previous determinations from the (pseudo)scalar sum rules [5–7, 9, 11–14, 23, 24, 31, 33, 50, 51, 53, 66, 67], the ones from e^+e^- [27, 50] and τ -decay [27, 29, 68] data and from nucleon and heavy-light sum rules [53] recently reviewed in [6, 50].

- Our bounds using Laplace sum rules including PT corrections to order α_s^4 are new and might be considered as alternatives of the existing bounds in the literature [1, 7, 9, 18, 66]. We plan to review these different determinations in a future publication [69].

References

- [1] C. Becchi, S. Narison, E. de Rafael and F.J. Yndurain, *Z. Phys.* **C8** (1981) 335.
- [2] E.G. Floratos, S. Narison and E. de Rafael, *Nucl. Phys.* **B155** (1979) 155.
- [3] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B147** (1979) 385, 448.
- [4] L.J. Reinders, H. Rubinstein and S. Yazaki, *Phys. Rept.* **127** (1985) 1.
- [5] S. Narison, *Riv.Nuovo Cim.* 10N2 (1987) 1.
- [6] For a review, see e.g.: S. Narison, *QCD as a theory of hadrons, Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol.* **17** (2002) 1 [hep-ph/0205006] and references therein.
- [7] For a review, see e.g.: S. Narison, *QCD spectral sum rules*, *World Sci. Lect. Notes Phys.* **26** (1989) 1 and references therein.
- [8] For reviews, see e.g.: S. Narison, *Phys. Rept.* **84** (1982) 263; S. Narison, *Acta Phys. Pol.* **B26** (1995) 68; S. Narison, hep-ph/9510270 (1995) and references therein.
- [9] S. Narison and E. de Rafael, *Phys. Lett.* **B103** (1981) 57.
- [10] S. Narison, *Phys. Lett.* **B104** (1981) 485.
- [11] S. Narison, N. Paver, E. de Rafael and D. Treleani, *Nucl. Phys.* **B212** (1983) 365.
- [12] S. Narison, N. Paver and D. Treleani, *Nuov. Cim.* **A74** (1983) 347.
- [13] S. Gorishny, A.L. Kataev and S.A. Larin, *Phys. Lett.* **B135** (1984) 457.

- [14] C.A. Dominguez and E. de Rafael, *Ann. Phys.* **174** (1987) 372.
- [15] D.J. Broadhurst, *Phys. Lett.* **B101** (1981) 423.
- [16] S. G. Gorishny, A. L. Kataev, S. A. Larin, and L. R. Surguladze, *Mod. Phys. Lett.* **A5** (1990) 2703.
- [17] K. G. Chetyrkin, *Phys. Lett.* **B390** (1997) 309.
- [18] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, *Phys. Rev. Lett.* **96** (2006) 012003.
- [19] E.V. Shuryak, *Rev. Mod. Phys.* **65** (1993) 1; T. Schafer and E.V. Shuryak, *Rev. Mod. Phys.* **70** (1998) 323.
- [20] B. V. Geshkeiben and B. L. Ioffe, *Nucl. Phys.* **B166** (1980) 340; B.L. Ioffe, K.N. Zybalyuk, *Eur. Phys. J.* **C27** (2003) 229; B. L. Ioffe, *Prog. Part. Nucl. Phys.* **56** (2006) 232 and references therein.
- [21] E. Gabrielli and P. Nason, *Phys. Lett.* **B313** (1993) 430; P. Nason and M. Porrati, *Nucl. Phys.* **B421** (1994) 518.
- [22] S. Narison and V. I. Zakharov, *Phys. Lett.* **B522** (2001) 266; Private communication from V. I. Zakharov.
- [23] J. Bijnens, J. Prades and E. de Rafael, *Phys. Lett.* **B348** (1995) 226.
- [24] C. A. Dominguez, L. Pirovano and K. Schilcher, *Phys. Lett.* **B425** (1998) 193.
- [25] E. Braaten, *Phys. Rev. Lett.* **60** (1988) 1606; S. Narison and A. Pich, *Phys. Lett.* **B211** (1988) 183.
- [26] E. Braaten, S. Narison and A. Pich, *Nucl. Phys.* **B 373**, 581 (1992).
- [27] S. Narison, *Phys. Lett.* **B358** (1995) 113; S. Narison, *Phys. Lett.* **B466** (1999) 35.
- [28] S. Narison, *Phys. Lett.* **B361** (1995) 121.
- [29] S. Narison, *Phys. Lett.* **B626** (2005) 101.
- [30] S. Narison, *Phys. Lett.* **B673** (2009) 30 and references therein.
- [31] K. Maltman and J. Kambor, *Phys. Rev.* **D65** (2002) 074013.
- [32] J.S. Bell and R.A. Bertlmann, *Nucl. Phys.* **B227** (1983) 435; R.A. Bertlmann, *Acta Phys. Austriaca* **53** (1981) 305.
- [33] K. G. Chetyrkin and A. Khodjamirian, *Eur. Phys. J.* **C46** (2006) 721.
- [34] S. Narison and R. Tarrach, *Phys. Lett.* **125B** (1983) 217.
- [35] V. P. Spiridonov and K. G. Chetyrkin, *Sov. J. Nucl. Phys.* **47** (1988) 522.
- [36] D.J. Broadhurst and S.C. Generalis, Open Univ. report, OUT-4102-12/R (1982), unpublished; *Phys. Lett.* **B139** (1984) 85; S.C. Generalis, Ph.D. thesis, Open Univ. report, OUT-4102-13 (1982), unpublished.
- [37] M. Jamin and M. Munz, *Z. Phys.* **C 66**, 633 (1995).
- [38] Y. Chung et al., *Z. Phys.* **C25** (1984) 151; H.G. Dosch, Non-Perturbative Methods (Montpellier 1985); H.G. Dosch, M. Jamin and S. Narison, *Phys. Lett.* **B220** (1989) 251.
- [39] B.L. Ioffe, *Nucl. Phys.* **B188** (1981) 317; B.L. Ioffe, **B191** (1981) 591; A.A.Ovchinnikov and A.A.Pivovarov, *Yad. Fiz.* **48** (1988) 1135.
- [40] S. Narison, *Phys. Lett.* **B605** (2005) 319.
- [41] G. Launer, S. Narison and R. Tarrach, *Z. Phys.* **C26** (1984) 433.
- [42] R. Akhoury and V.I. Zakharov, *Phys. Lett.* **B 438** (1998) 165.
- [43] K. Chetyrkin, S. Narison and V.I. Zakharov, *Nucl. Phys.* **B 550**, 353 (1999).
- [44] O. Andreev and V.I. Zakharov, *Phys. Rev.* **D76** (2007) 047705; F. Jugeau, S. Narison, H. Ratsimbarison, *Phys. Lett.* **B722** (2013) 111.
- [45] S. Narison and V.I. Zakharov, *Phys.Lett.* **B679** (2009) 355.
- [46] P.E.L. Rakow, *PoS LAT2005* (2006) 284; R. Horsley, P.E.L. Rakow, G. Schierholz, *Nucl. Phys. Proc. Suppl.* **106** (2002) 870; A. Di Giacomo, G.C. Rossi, *Phys. Lett.* **B100** (1981) 481; G. S. Bali, C. Bauer and A. Pineda, arXiv: 1403.6477v1[hep-ph] (2014); T. Lee, *Phys. Rev.* **D82** (2010) 114021.
- [47] S. Narison, *Phys. Lett.* **B300** (1993) 293.
- [48] S. Narison, *Phys. Lett.* **B693** (2010) 559; Erratum *ibid* 705 (2011) 544; *ibid*, *Phys. Lett.* **B706** (2012) 412; *ibid*, *Phys. Lett.* **B707** (2012) 259.
- [49] M.A. Shifman, *Nucl. Phys. Proc. Suppl.* **207-208** (2010) 298 and references therein.
- [50] For reviews, see e.g.: S. Narison, *Phys.Rev.* **D74** (2006) 034013; S. Narison, hep-ph/0202200 (section of [6]); S. Narison, *Nucl. Phys. Proc. Suppl.* **86** (2000) 242 and references quoted therein.
- [51] Particle Data Group, J. Beringer et al. (Particle Data Group), *Phys. Rev.* **D86**, 010001 (2012) and references therein.
- [52] S. Bethke, *Nucl. Phys. Proc. Suppl.* **234** (2013) 229 and references therein.
- [53] H.G. Dosch and S. Narison, *Phys. Lett.* **B417** (1998) 173; S. Narison, *Phys. Lett.* **B216** (1989) 191.
- [54] R.M. Albuquerque, S. Narison, *Phys. Lett.* **B694** (2010) 217; R.M. Albuquerque, S. Narison, M. Nielsen, *Phys. Lett.* **B684** (2010) 236.

- [55] K.G. Chetyrkin, J.H. Kühn and M. Steinhauser, *Comput. Phys. Comm* **133** (2000) 43 and references therein.
- [56] R.A. Bertlmann, G. Launer and E. de Rafael, *Nucl. Phys.* **B250** (1985) 61; R.A. Bertlmann et al., *Z. Phys.* **C39** (1988) 231; S. Bodenstein et al., *JHEP* **1201** (2012) 093015.
- [57] F.J. Yndurain, hep-ph/9903457.
- [58] S. Narison, *Phys. Lett.* **B387** (1996) 162.
- [59] R.A. Bertlmann and H. Neufeld, *Z. Phys.* **C27** (1985) 437.
- [60] S. Narison, *Phys. Lett.* **B624** (2005) 223.
- [61] D. Boito et al., *Phys. Rev.* **D85** (2012) 093015 and references therein.
- [62] C. McNeile et al., *Phys. Rev.* **87** (2013) 034503.
- [63] J. Rosner and S.L. Stone in Particle Data Group, J. Beringer et al. (Particle Data Group), *Phys. Rev.* **D86**, 010001 (2012).
- [64] S. Aoki et al., arXiv:1310.8555 [hep-lat] (2013) and references therein.
- [65] D. B. Kaplan and A. V. Manohar, *Phys.Rev.Lett.* **56** (1986) 2004; H. Pagels and A. Zepeda, *Phys. Rev.* **D5** (1972) 3262;
- [66] L. Lellouch, E. de Rafael and J. Taron, *Phys. Lett.* **B414** (1997) 195.
- [67] P. Colangelo et al., *Phys. Lett.* **B408** (1997) 340; M. Jamin, S. Oller and A. Pich, *Phys. Rev.* **D74** (2006) 074009; S. Bodenstein et al., *JHEP* **1307** (2013) 138.
- [68] E. Gamiz et al., *Phys. Rev. Lett.* **94** (2005) 011803.
- [69] S. Narison, talk presented at QCD 14, 30 june - 4 july 2014, Montpellier.